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HOMOGENEOUS CODIMENSION ONE FOLIATIONS ON NONCOMPACT SYMMETRIC SPACES

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Abstract

We determine the isometric congruence classes of homogeneous Riemannian foliations of codimension one on connected irreducible Riemannian symmetric spaces of noncompact type. As an application we show that on each connected irreducible Riemannian symmetric space of noncompact type and rank greater than two there exist noncongruent homogeneous isoparametric systems with the same principal curvatures, counted with multiplicities.

1. Introduction

An action of a Lie group on a manifold is of cohomogeneity one if there exists an orbit of codimension one. Our motivation for this work is the classical problem: *For a connected complete Riemannian manifold M, determine the space $\mathfrak{M}$ of all isometric cohomogeneity one actions on M modulo orbit equivalence.* Two cohomogeneity one actions on M are orbit equivalent if there exists an isometry of M mapping the orbits of one of these actions onto the orbits of the other action.

For the sphere S^n with its standard metric Hsiang and Lawson [12] proved that every element in $\mathfrak{M}$ can be represented by the isotropy representation of an $(n+1)$-dimensional Riemannian symmetric space of rank two. In successive work by Takagi [25], Uchida [27], D'Atri [8], Iwata [13, 14] and Kollross [16] the problem was solved for all connected, simply connected, irreducible Riemannian symmetric spaces of compact type. For some exceptional symmetric spaces the moduli space $\mathfrak{M}$ is an empty set, otherwise $\mathfrak{M}$ is a finite set and representatives for all

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actions are explicitly known. The main tool for proving these results is the classification of maximal closed connected subgroups of semisimple compact Lie groups.

For noncompact manifolds the methods employed by the above authors are not applicable. A major difficulty in the noncompact case arises from the fact that there can exist large families of nonconjugate subgroups of the isometry group having the same orbits. In general it is a difficult problem to decide about orbit equivalence for nonconjugate subgroups. The aim of this paper is to present a partial solution to this problem for Riemannian symmetric spaces of noncompact type.

Let M be a connected irreducible Riemannian symmetric spaces of noncompact type. General theory about cohomogeneity one actions (see e.g., [20] for the compact case and [3] for the general case) implies that any such action on M either induces a foliation on M or has exactly one singular orbit. This induces a disjoint union $\mathfrak{M} = \mathfrak{M}_F \cup \mathfrak{M}_S$, where $\mathfrak{M}_F$ is the set of all homogeneous codimension one foliations on M modulo isometric congruence and $\mathfrak{M}_S$ is the set of all connected normal homogeneous submanifolds with codimension ≥ 2 in M modulo isometric congruence. The main result of this paper is a complete description of $\mathfrak{M}_F$.

Denote by r the rank of M and by $\mathbb{R}P^{r-1}$ the $(r-1)$-dimensional real projective space. To each $\ell \in \mathbb{R}P^{r-1}$ we associate a homogeneous codimension one foliation $\mathfrak{F}_\ell$ on M all of whose leaves are isometrically congruent to each other, and to each $i \in \{1, \ldots, r\}$ we associate a homogeneous codimension one foliation $\mathfrak{F}_i$ on M that has exactly one minimal leaf. The symmetry group $\mathrm{Aut}(DD)$ of the Dynkin diagram of the restricted root system Σ associated to M acts on a set of simple roots in Σ, which is a set of r elements and forms a basis of an r-dimensional Euclidean vector space. This action then induces canonically an action on $\mathbb{R}P^{r-1}$, and thus we get an action of the finite symmetry group $\mathrm{Aut}(DD)$ on $\mathbb{R}P^{r-1} \cup \{1, \ldots, r\}$.

Theorem. *Let M be a connected irreducible Riemannian symmetric space of noncompact type and with rank r. The moduli space $\mathfrak{M}_F$ of all noncongruent homogeneous codimension one foliations on M is isomorphic to the orbit space of the action of* $\mathrm{Aut}(DD)$ *on* $\mathbb{R}P^{r-1} \cup \{1, \ldots, r\}$*:*

$$\mathfrak{M}_F \cong (\mathbb{R}P^{r-1} \cup \{1, \ldots, r\})/\mathrm{Aut}(DD).$$

A remarkable consequence is that $\mathfrak{M}_F$ depends only on the rank and on possible duality or triality principles on the symmetric space. For instance, for $\mathrm{SO}(17,\mathbb{C})/\mathrm{SO}(17)$, $\mathrm{Sp}(8,\mathbb{R})/\mathrm{U}(8)$, $\mathrm{Sp}(8,\mathbb{C})/\mathrm{Sp}(8)$, $\mathrm{SO}(16,\mathbb{H})/\mathrm{U}(16)$, $\mathrm{SO}(17,\mathbb{H})/\mathrm{U}(17)$, $E_8^8/\mathrm{SO}(16)$, $E_8^{\mathbb{C}}/E_8$ and for the hyperbolic Grassmannians $G_8^*(\mathbb{R}^{n+16})$ $(n \geq 1)$, $G_8^*(\mathbb{C}^{n+16})$ $(n \geq 0)$, $G_8^*(\mathbb{H}^{n+16})$ $(n \geq 0)$ the moduli space $\mathfrak{M}_F$ of all noncongruent homogeneous codimension one foliations is isomorphic to $\mathbb{R}P^7 \cup \{1,\ldots,8\}$. Another consequence is that on each of the hyperbolic spaces $\mathbb{R}H^n$, $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$ there exist exactly two congruence classes of homogeneous codimension one foliations. One of them is given by the horosphere foliation, the other is not so well-known and has been studied in more detail by the first author in [4].

The construction of the model foliations $\mathfrak{F}_\ell$ and $\mathfrak{F}_i$ is in fact elementary, they arise as orbits of codimension one subgroups of the solvable group in an Iwasawa decomposition of the isometry group. These model foliations have quite interesting geometric features. In the rank one case $\mathfrak{F}_\ell$ is a foliation by horospheres. For general rank the leaves of $\mathfrak{F}_\ell$ are still all congruent to each other (Proposition 3.1). If $r \geq 2$, there exists an $(r-2)$-dimensional family of harmonic foliations (i.e., all leaves are minimal) among the model foliations $\mathfrak{F}_\ell$, and hence the corresponding projections $F_\ell : M \to \mathbb{R}$ onto the space of leaves of $\mathfrak{F}_\ell$ are harmonic functions (Corollary 3.2). Each foliation $\mathfrak{F}_i$ has exactly one minimal leaf (Corollary 4.5). Moreover, there exists a reflection of M in a totally geodesic submanifold leaving the unique minimal leaf invariant and interchanging the two leaves at any given positive distance from the minimal leaf (Proposition 4.6). A mean curvature argument shows that two such foliations are not isometrically congruent when they come from simple roots of different length. If α_i and α_j are simple roots with the same length, then there exists a bijective correspondence between the leaves of $\mathfrak{F}_i$ and $\mathfrak{F}_j$ such that corresponding leaves have the same principal curvatures, counted with multiplicities (Theorem 4.7). Moreover, we prove that $\mathfrak{F}_i$ and $\mathfrak{F}_j$ are congruent if and only if α_i and α_j are related by a Dynkin diagram symmetry, which implies (Corollary 4.9): *On every connected irreducible Riemannian symmetric space of noncompact type and with rank* $r \geq 3$ *there exist noncongruent homogeneous isoparametric systems for which corresponding leaves have the same principal curvatures, counted with multiplicities.* Among the inhomogeneous isoparametric systems on spheres constructed by Ferus, Karcher and Münzner [9] from Clifford modules one can find systems with such an isospectral feature. The above examples show that this curious isospectral phenomenon also occurs within the class of homogeneous isoparametric systems.

The parameter space for the model foliations is $\mathbb{R}P^{r-1} \cup \{1, \ldots, r\}$. The two major problems we have to solve is to determine the congruency classes among these model foliations and to show that any homogeneous codimension one foliation on M is congruent to one of these model foliations. The above mentioned geometric investigations provide partial answers to the first problem, but the result about the noncongruent homogeneous isoparametric systems indicates that it is impossible to solve it geometrically. We apply a conjugacy result of Alekseevsky [1] about completely solvable transitive groups of motions to reformulate the first problem in algebraic terms. One of the crucial steps in solving this algebraic problem is a rigidity result for Lie algebra isomorphisms of the nilpotent Lie algebra in an Iwasawa decomposition of $\mathfrak{g}$ (Theorem 3.4). This result, together with some structure theory for the solvable Lie algebras $\mathfrak{s}_\ell$, enables us to decide the congruency problem for the foliations $\mathfrak{F}_\ell$ (Theorem 3.5). For the foliations $\mathfrak{F}_i$ the algebraic problem is quite easy to handle when the multiplicity of the simple root is greater than one. In case of multiplicity one we introduce certain diagrams that allow us to deal with the congruency question in a similar fashion as Dynkin diagrams are used for semisimple complex Lie algebras (Theorem 4.8).

In order to prove that there are no other homogeneous codimension one foliations up to orbit equivalence we proceed as follows. First, using the Levi decomposition of a Lie group and a structure theorem by Malcev [19] about solvable Lie groups, we can restrict to connected solvable subgroups S of G^o acting on M freely and with cohomogeneity one. The Lie algebra $\mathfrak{s}$ of such a Lie group S is contained in a maximal solvable subalgebra of $\mathfrak{g}$ that contains, by a result of Mostow [21], a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We prove that $\mathfrak{h}$ must be a maximally noncompact Cartan subalgebra of $\mathfrak{g}$, which implies that $\mathfrak{s} \subset \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$ for some suitable Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ of $\mathfrak{g}$ and a maximal abelian subalgebra $\mathfrak{t}$ in the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ (Proposition 5.2). The orthogonal projection $\mathfrak{s}_n$ of $\mathfrak{s}$ onto $\mathfrak{a} + \mathfrak{n}$ is a Lie subalgebra of $\mathfrak{a} + \mathfrak{n}$ (Proposition 5.4). We classify all possible Lie subalgebras of $\mathfrak{a} + \mathfrak{n}$ with codimension one (Lemma 5.3), which leads, up to orbit equivalence, to the model foliations constructed above. The final step is then to prove that S and the connected subgroup of AN with Lie algebra $\mathfrak{s}_n$ induce isometrically congruent foliations (Theorem 5.5).

The paper is organized as follows. In the next section we present some basic material about the restricted root systems associated to Riemannian symmetric spaces of noncompact type. In Section 3 we construct the foliations $\mathfrak{F}_\ell$ and study the congruency problem for them, and in Section 4 we deal with the analogue for the foliations $\mathfrak{F}_i$. In Section 5 we prove that any homogeneous codimension one foliation on M is isometrically congruent to one of the foliations constructed in Sections 3 and 4.

We mention here that for the standard spaces of nonpositive constant curvature the relatively simple structure of the Gauss-Codazzi equations for submanifolds can be used to determine the entire moduli space $\mathfrak{M}$. For the Euclidean space $\mathbb{R}^n$ it follows from work of Levi Civita [18] (for $n = 3$) and Segre [23] (in general) on isoparametric hypersurfaces that $\mathfrak{M}$ is a set of n points, one of which represents the subset $\mathfrak{M}_F$. Analogous work of E. Cartan [7] for the real hyperbolic space $\mathbb{R}H^n$ shows that for this space the moduli space $\mathfrak{M}$ consists of $n+1$ points, two of which represent the subset $\mathfrak{M}_F$. An analogous approach for more general manifolds leads to hopeless calculations. We finally remark that the moduli space $\mathfrak{M}_S$ is not known for any connected irreducible Riemannian symmetric space of noncompact type different from $\mathbb{R}H^n$. Some recent work of the first author and Brück [5] on cohomogeneity one actions on the hyperbolic spaces $\mathbb{C}H^n$, $\mathbb{H}H^n$ and $\mathbb{O}H^2$ indicates that the structure of this moduli space is far more complicated. In [6] the authors determined the subset of $\mathfrak{M}_S$ that corresponds to totally geodesic singular orbits.

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2. Preliminaries

For details about symmetric spaces and structure theory of semisimple Lie algebras we refer to the monographs [10], [11], [17] and [22]. For root systems Σ and corresponding sets $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots we adopt the notation from [17].

Let M be a connected irreducible Riemannian symmetric space of noncompact type, $n = \dim M$, r the rank of M, and denote by G the isometry group of M and by G^o its identity component. Let $o \in M$ and K resp. K^o the isotropy subgroup of G resp. G^o at o. As M is simply connected, K^o is the identity component of K. We denote by $\mathfrak{g}$ and $\mathfrak{k}$

the Lie algebra of G and K, respectively, and by B the Killing form of $\mathfrak{g}$. If $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. We identify $\mathfrak{p}$ with T_oM in the usual manner. If $\theta : \mathfrak{g} \to \mathfrak{g}$ is the corresponding Cartan involution, we get a positive definite inner product on $\mathfrak{g}$ by $\langle X, Y\rangle = -B(X, \theta Y)$ for all $X, Y \in \mathfrak{g}$. We normalize the Riemannian metric on M such that its restriction to $T_oM \times T_oM = \mathfrak{p} \times \mathfrak{p}$ coincides with $\langle\cdot,\cdot\rangle$.

Let $\mathfrak{a}$ be a maximal abelian subspace in $\mathfrak{p}$ and denote by $\mathfrak{a}^*$ the dual vector space of $\mathfrak{a}$. For each $\lambda \in \mathfrak{a}^*$ we define $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid \mathrm{ad}(H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}\}$. Since the linear transformations $\mathrm{ad}(H) : \mathfrak{g} \to \mathfrak{g}$, $H \in \mathfrak{a}$, form a commuting family of selfadjoint transformations on $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$, they induce an eigenspace decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda$, the so-called restricted root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Here, Σ is the set of all nonzero $\lambda \in \mathfrak{a}^*$ for which $\mathfrak{g}_\lambda$ is nontrivial. Each $\lambda \in \Sigma$ is called a restricted root and the corresponding eigenspace $\mathfrak{g}_\lambda$ is called a restricted root space. The eigenspace $\mathfrak{g}_0$ with respect to $0 \in \mathfrak{a}^*$ is given by $\mathfrak{g}_0 = C(\mathfrak{a};\mathfrak{k}) \oplus \mathfrak{a}$, where $C(\mathfrak{a};\mathfrak{k})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The root system Σ is either reduced and then of type $A_r, B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2$ or nonreduced and then of type BC_r.

For each $\lambda \in \mathfrak{a}^*$ let $H_\lambda \in \mathfrak{a}$ be the dual vector in $\mathfrak{a}$ with respect to the Killing form, that is, $\lambda(H) = \langle H_\lambda, H\rangle$ for all $H \in \mathfrak{a}$. Then we get an inner product on $\mathfrak{a}^*$, which we also denote by $\langle\cdot,\cdot\rangle$, by means of $\langle\lambda,\mu\rangle = \langle H_\lambda, H_\mu\rangle$ for all $\lambda, \mu \in \mathfrak{a}^*$. We choose a set $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$ of simple roots in Σ and denote the resulting set of positive restricted roots by Σ^+. Each root $\lambda \in \Sigma$ can be written as $\lambda = c_1\alpha_1 + \cdots + c_r\alpha_r$ with all integers c_k either all ≥ 0 or all ≤ 0. The sum $c_1 + \cdots + c_r$ is called the level of the root λ. Let m be the level of the maximal root in Σ^+, that is, $m = m_1 + \cdots + m_r$ such that $m_1\alpha_1 + \cdots + m_r\alpha_r$ is the maximal root in Σ^+.

By $\mathrm{Aut}(DD)$ we denote the group of symmetries of the Dynkin diagram associated to Λ. Each symmetry $P \in \mathrm{Aut}(DD)$ can be linearly extended to a linear isometry of $\mathfrak{a}^*$, which we also denote by P. Denote by Φ the linear isometry from $\mathfrak{a}^*$ to $\mathfrak{a}$ defined by $\Phi(\lambda) = H_\lambda$ for all $\lambda \in \mathfrak{a}^*$. Then $\widetilde{P} = \Phi \circ P \circ \Phi^{-1}$ is a linear isometry of $\mathfrak{a}$ with $\widetilde{P}(H_\lambda) = H_\mu$ if and only if $P(\lambda) = \mu$, $\lambda, \mu \in \mathfrak{a}^*$. Since P is an orthogonal transformation, $\widetilde{P}$ is just the dual map of $P^{-1} : \mathfrak{a}^* \to \mathfrak{a}^*$. In this way each symmetry $P \in \mathrm{Aut}(DD)$ induces linear isometries of $\mathfrak{a}^*$ and $\mathfrak{a}$, both of which we will denote by P, since it will always be clear from the context which of these two we are using.

The choice of Λ induces a gradation $\mathfrak{g} = \bigoplus_{k=-m}^{m} \mathfrak{g}^k$ of $\mathfrak{g}$, where $\mathfrak{g}^k$ is the linear subspace of $\mathfrak{g}$ spanned by all root spaces corresponding to roots of level $k \in \{-m, \dots, m\}$. This gradation is of type α_0 (see [15]), i.e., $\mathfrak{g}^1$ generates the subalgebra $\bigoplus_{k=1}^{m} \mathfrak{g}^k$ and $\mathfrak{g}^{-1}$ generates the subalgebra $\bigoplus_{k=-m}^{-1} \mathfrak{g}^k$.

For $\lambda, \mu \in \Sigma^+$, λ and μ linearly independent, the μ-string containing λ is the sequence $\lambda - p\mu, \dots, \lambda, \dots, \lambda + q\mu$ such that p and q are nonnegative integers, $\lambda - p\mu, \dots, \lambda + q\mu \in \Sigma^+$, and $\lambda - (p+1)\mu, \lambda + (q+1)\mu \notin \Sigma^+$. Then we have $2\langle \lambda, \mu \rangle / \langle \mu, \mu \rangle = p - q$. The integer $p + q \in \{0, 1, 2, 3\}$ is called the length of the μ-string containing λ. It is worthwhile to point out at this point that this formula enables us to calculate the inner product between roots by using the length of strings involving them. In particular, the Dynkin diagram of the restricted root system can be constructed from the set Λ of simple roots just from the string relations between the simple roots. More precisely, assume that α and β are two simple roots and consider the α-string containing β and the β-string containing α, that is, $\beta, \dots, \beta + q_{\beta\alpha}\alpha$ and $\alpha, \dots, \alpha + q_{\alpha\beta}\beta$. If $q_{\alpha\beta} = q_{\beta\alpha}$, then connect α and β by $q_{\alpha\beta}$ lines. If $q_{\alpha\beta} \neq q_{\beta\alpha}$, then connect α and β by $\max\{q_{\alpha\beta}, q_{\beta\alpha}\}$ lines and draw an arrow from α to β if $q_{\alpha\beta} > q_{\beta\alpha}$ and from β to α if $q_{\beta\alpha} > q_{\alpha\beta}$.

If we define $\mathfrak{n} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda$ we get an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of $\mathfrak{g}$. The subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ is an m-step nilpotent subalgebra, where m is the level of the maximal root in Σ^+. The Lie algebra $\mathfrak{n}$ has a natural gradation $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_m$, where $\mathfrak{n}_k = \mathfrak{g}^k$. Note that $\mathfrak{n}_1$ generates $\mathfrak{n}$. Moreover, $\mathfrak{a} + \mathfrak{n}$ is a solvable subalgebra of $\mathfrak{g}$ with $[\mathfrak{a} + \mathfrak{n}, \mathfrak{a} + \mathfrak{n}] = \mathfrak{n}$. The connected subgroups A, N, AN of G with Lie algebras $\mathfrak{a}, \mathfrak{n}, \mathfrak{a} + \mathfrak{n}$, respectively, are simply connected and AN acts simply transitively on M. The symmetric space M is isometric to the connected, simply connected, solvable Lie group AN equipped with the left-invariant Riemannian metric that is induced from the inner product $\langle \cdot, \cdot \rangle$. We denote by ∇ the Levi Civita connection of the solvable Lie group AN equipped with this left-invariant metric. Using the Koszul formula for the Levi Civita connection, and using the fact that the metric on AN is left-invariant, we get $2\langle \nabla_X Y, Z \rangle = \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle$ for all $X, Y, Z \in \mathfrak{a} + \mathfrak{n}$, where we regard elements in the Lie algebra $\mathfrak{a} + \mathfrak{n}$ as left-invariant vector fields on AN.

3. The foliations $\mathfrak{F}_\ell$

Let ℓ be a linear line in $\mathfrak{a}$. Then the orthogonal complement $\mathfrak{s}_\ell = (\mathfrak{a}+\mathfrak{n})\ominus\ell = (\mathfrak{a}\ominus\ell)+\mathfrak{n}$ of ℓ in $\mathfrak{a}+\mathfrak{n}$ is a subalgebra of $\mathfrak{a}+\mathfrak{n}$ of codimension one. Let S_ℓ be the connected Lie subgroup of AN with Lie algebra $\mathfrak{s}_\ell$. Then the orbits of the action of S_ℓ on M form a homogeneous foliation $\mathfrak{F}_\ell$ on M of codimension one.

Let $H_\ell \in \mathfrak{a}$ be a unit vector such that $\ell = \mathbb{R}H_\ell$. From the above expression for the Levi Civita connection we get $\nabla_{H_\ell}H_\ell = 0$. General theory about Riemannian foliations implies that the integral curves of H_ℓ are geodesics in AN and hence intersect each leaf of $\mathfrak{F}_\ell$, and at the points of intersection it intersects perpendicularly. Let $\gamma : \mathbb{R} \to AN$ be the geodesic in AN with $\gamma(0) = o$ and $\dot\gamma(0) = H_\ell$. Then $\gamma(\mathbb{R}) \subset A$, and γ intersects each leaf of $\mathfrak{F}_\ell$. Moreover, as A is abelian, $N \subset S_\ell$ and $AN = NA$, we get $\gamma(t)S_\ell = S_\ell\gamma(t)$ and hence $S_\ell\cdot\gamma(t) = \gamma(t)(\gamma(t)^{-1}S_\ell\gamma(t))\cdot o = \gamma(t)S_\ell \cdot o$ for all $t \in \mathbb{R}$. This shows that each leaf of $\mathfrak{F}_\ell$ is obtained by a suitable left translation of $S_\ell \cdot o$ in AN. In particular, all leaves of $\mathfrak{F}_\ell$ are congruent to each other under an isometry of M. In order to study the geometry of the foliation $\mathfrak{F}_\ell$ it is therefore sufficient to study the geometry of the leaf $S_\ell \cdot o$.

The vector H_ℓ is a unit normal vector of $S_\ell \cdot o$ at o. We denote by A_{H_ℓ} the shape operator of $S_\ell \cdot o$ at o with respect to H_ℓ and by h the second fundamental form of $S_\ell \cdot o$. Since $\mathrm{ad}(H_\ell)$ is a selfadjoint endomorphism on $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle$, the above expression for the Levi Civita connection and the Weingarten formula imply $\langle h(X,Y), H_\ell\rangle = \langle A_{H_\ell}X, Y\rangle = \langle \mathrm{ad}(H_\ell)X, Y\rangle$ for all $X, Y \in \mathfrak{s}_\ell = T_o(S_\ell \cdot o)$. Therefore $A_{H_\ell} = \mathrm{ad}(H_\ell)|\mathfrak{s}_\ell$, and we have proved:

Proposition 3.1. *Using the above notations, we have:*

(1) *All leaves of $\mathfrak{F}_\ell$ are isometrically congruent to each other.*

(2) *The shape operator A_{H_ℓ} of the leaf $S_\ell \cdot o$ of $\mathfrak{F}_\ell$ through o is given by $A_{H_\ell} = \mathrm{ad}(H_\ell)|\mathfrak{s}_\ell$.*

(3) *The subspace $\mathfrak{a} \ominus \ell$ is a principal curvature space of $S_\ell \cdot o$ with corresponding principal curvature 0, and for each $\lambda \in \Sigma^+$ the root space $\mathfrak{g}_\lambda$ is a principal curvature space of $S_\ell \cdot o$ with corresponding principal curvature $\lambda(H_\ell)$.*

(4) *For the (constant) mean curvature* μ_ℓ *of each leaf of* $\mathfrak{F}_\ell$ *we have*

$$\mu_\ell = \frac{1}{n-1} \sum_{\lambda \in \Sigma^+} (\dim \mathfrak{g}_\lambda) \lambda(H_\ell).$$

A more general treatment of foliations induced by subalgebras on Lie groups with left-invariant Riemannian metrics can be found in [26]. It is worthwhile to mention the following observation. If the rank of M is greater than one, then the unit sphere in $\mathfrak{a}$ is connected. If H_ℓ lies in the positive Weyl chamber C^+ of $\mathfrak{a}$, then the mean curvature of $S_\ell \cdot o$ is positive, whereas it is negative when we choose H_ℓ in the negative Weyl chamber C^- of $\mathfrak{a}$. For continuity reasons we therefore find on each great circle in the unit sphere in $\mathfrak{a}$ through $H_\ell \in C^+$ and $-H_\ell \in C^-$ a point $H_{\ell'} \in \mathfrak{a}$ such that the corresponding orbit $S_{\ell'} \cdot o$ is minimal. Since all leaves of $\mathfrak{F}_{\ell'}$ are isometrically congruent to each other we conclude:

Corollary 3.2. *On each connected irreducible Riemannian symmetric space of noncompact type and rank* $r \geq 2$ *there exists an* $(r-2)$*-dimensional family of homogeneous harmonic foliations of codimension one.*

A foliation is called harmonic if all its leaves are minimal submanifolds. It is known that a foliation on a Riemannian manifold is harmonic if and only if the canonical projection from the manifold onto the space of leaves of the foliation is a harmonic map. From the above corollary we thus get an $(r-2)$-dimensional family of harmonic functions on the symmetric space M with the property that its level sets are homogeneous hypersurfaces.

Our next aim is to investigate when two such foliations $\mathfrak{F}_\ell$ and $\mathfrak{F}_{\ell'}$ are isometrically congruent. We start with some basic properties about the subalgebras $\mathfrak{s}_\ell$.

Lemma 3.3. *Let* $r \geq 2$*. The following statements hold:*

(1) *The Lie algebra* $\mathfrak{s}_\ell$ *is completely solvable, that is, there exists a basis of* $\mathfrak{s}_\ell$ *such that* $\mathrm{ad}(\mathfrak{s}_\ell)$ *consists of upper triangular matrices with respect to that basis.*

(2) *The subalgebra*

$$\mathfrak{c}_\ell = \begin{cases} (\mathfrak{a} \ominus \ell) \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_{2\lambda} & \text{if } \ell = \mathbb{R}H_\lambda \text{ for some } \lambda \in \Sigma^+,\ \lambda/2 \notin \Sigma^+ \\ \mathfrak{a} \ominus \ell & \text{otherwise} \end{cases}$$

is a Cartan subalgebra of $\mathfrak{s}_\ell$*. Note that* $\mathfrak{g}_{2\lambda}$ *is trivial if* $2\lambda \notin \Sigma^+$*.*

(3) *The set* $\{Y \in \mathfrak{s}_\ell \mid \mathrm{ad}(Y)^k = 0 \text{ for some } k \in \mathbb{N}\}$ *of all nilpotent elements in* $\mathfrak{s}_\ell$ *is equal to* $\mathfrak{n}$.

Proof. (1): Consider the gradation $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_m$ of $\mathfrak{s}_\ell$. Choose a basis of $\mathfrak{s}_\ell$ such that the first vectors in that basis are in $\mathfrak{n}_m$, the next ones in $\mathfrak{n}_{m-1}$, and so on, and the last vectors are in $\mathfrak{a} \ominus \ell$. It then follows immediately that $\mathrm{ad}(\mathfrak{s}_\ell)$ consists of upper triangular matrices with respect to that basis.

(2): We first assume that $\ell = \mathbb{R}H_\lambda$ for some positive root $\lambda \in \Sigma^+$ with $\lambda/2 \notin \Sigma^+$. It is clear that $\mathfrak{c}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_{2\lambda}$ is a nilpotent subalgebra of $\mathfrak{s}_\ell$. If in particular $\mathfrak{g}_{2\lambda}$ is trivial, then $\mathfrak{c}_\ell$ is an abelian subalgebra of $\mathfrak{s}_\ell$. We have to show that $\mathfrak{c}_\ell$ equals its own normalizer in $\mathfrak{s}_\ell$. Let X be in the normalizer of $\mathfrak{c}_\ell$ in $\mathfrak{s}_\ell$. We write $X = X_1 + X_2$ with $X_1 \in \mathfrak{c}_\ell$ and $X_2 \in \mathfrak{n} \ominus (\mathfrak{g}_\lambda \oplus \mathfrak{g}_{2\lambda}) = \mathfrak{s}_\ell \ominus \mathfrak{c}_\ell$. Since $[\mathfrak{c}_\ell, \mathfrak{c}_\ell] \subset \mathfrak{c}_\ell$ and $[\mathfrak{c}_\ell, \mathfrak{s}_\ell \ominus \mathfrak{c}_\ell] \subset \mathfrak{s}_\ell \ominus \mathfrak{c}_\ell$ we have $[\mathfrak{c}_\ell, X_2] = 0$. We now decompose X_2 into $X_2 = \sum_{\mu \in \Sigma^+} X_2^\mu$. If $X_2 \neq 0$, there exists a root $\nu \in \Sigma^+$ such that $X_2^\nu \neq 0$ and the level of ν is less or equal than the level of all roots $\mu \in \Sigma^+$ for which $X_2^\mu \neq 0$. The orthogonal projection of $[\mathfrak{c}_\ell, X_2]$ onto $\mathfrak{g}_\nu$ is equal to the orthogonal projection of $[\mathfrak{a} \ominus \ell, X_2]$ onto $\mathfrak{g}_\nu$. Since $\nu \notin \{\lambda, 2\lambda\}$, and hence $H_\nu \notin \mathbb{R}H_\lambda$, the orthogonal projection of $[\mathfrak{a} \ominus \ell, X_2]$ onto $\mathfrak{g}_\nu$ is nontrivial, which contradicts $[\mathfrak{c}_\ell, X_2] = 0$. Thus we must have $X_2 = 0$, and it follows that $\mathfrak{c}_\ell$ is a Cartan subalgebra of $\mathfrak{s}_\ell$.

Next, assume that $\ell \neq \mathbb{R}H_\lambda$ for all $\lambda \in \Sigma^+$. We have to show that the normalizer of the abelian subalgebra $\mathfrak{c}_\ell = \mathfrak{a} \ominus \ell$ in $\mathfrak{s}_\ell$ is equal to $\mathfrak{c}_\ell$. Let X be in that normalizer and write $X = X_1 + X_2$ with $X_1 \in \mathfrak{c}_\ell$ and $X_2 \in \mathfrak{n} = \mathfrak{s}_\ell \ominus \mathfrak{c}_\ell$. We now decompose X_2 into $X_2 = \sum_{\mu \in \Sigma^+} X_2^\mu$. Then $[H, X] = [H, X_2] = \sum_{\mu \in \Sigma^+} \mu(H) X_2^\mu \in \mathfrak{c}_\ell = \mathfrak{a} \ominus \ell$ for all $H \in \mathfrak{a} \ominus \ell$. Since by assumption H_ℓ is not a multiple of any root vector H_λ, we have $\mu(\mathfrak{a} \ominus \ell) \neq 0$ for all $\mu \in \Sigma^+$, which implies $X_2^\mu = 0$ for all $\mu \in \Sigma^+$. Thus $X_2 = 0$, and it follows that $\mathfrak{c}_\ell = \mathfrak{a} \ominus \ell$ is a Cartan subalgebra of $\mathfrak{s}_\ell$.

(3): From the graded Lie algebra structure of $\mathfrak{n}$ it is clear that every element in $\mathfrak{n}$ is nilpotent. Let $0 \neq H \in \mathfrak{a} \ominus \ell$ and $X \in \mathfrak{n}$. Then there exists a simple root $\alpha \in \Lambda$ such that $\alpha(H) \neq 0$. Let $Y \in \mathfrak{g}_\alpha$ be a nonzero vector. We will now show by induction that $(\mathrm{ad}(H + X)^k Y)_{\mathfrak{g}_\alpha} = \alpha(H)^k Y \neq 0$ for all positive integers k, where $(\cdot)_{\mathfrak{g}_\alpha}$ denotes the orthogonal projection onto $\mathfrak{g}_\alpha$.

Since $X \in \mathfrak{n}$ and $Y \in \mathfrak{g}_\alpha$, and as α is a simple root, we have $[X, Y] \in \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_m$, and hence the $\mathfrak{g}_\alpha$-component of $\mathrm{ad}(X)Y$ vanishes. It follows that $(\mathrm{ad}(H + X)Y)_{\mathfrak{g}_\alpha} = (\mathrm{ad}(H)Y)_{\mathfrak{g}_\alpha} = \alpha(H)Y \neq 0$, which proves the assertion for $k = 1$.

Now assume that the assertion holds for the positive integer k and consider the expression $(\mathrm{ad}(H+X)^{k+1}Y)_{\mathfrak{g}_\alpha} = (\mathrm{ad}(H+X)\mathrm{ad}(H+X)^kY)_{\mathfrak{g}_\alpha}$. Since $\mathrm{ad}(H+X)^kY \in \mathfrak{n}$, only the $\mathfrak{g}_\alpha$-component of it contributes to a $\mathfrak{g}_\alpha$-component in $(\mathrm{ad}(H+X)\mathrm{ad}(H+X)^kY)_{\mathfrak{g}_\alpha}$, that is, $(\mathrm{ad}(H+X)^{k+1}Y)_{\mathfrak{g}_\alpha} = (\mathrm{ad}(H+X)((\mathrm{ad}(H+X)^k)Y)_{\mathfrak{g}_\alpha})_{\mathfrak{g}_\alpha}$. The assumption and the statement for $k=1$ then imply $(\mathrm{ad}(H+X)^{k+1}Y)_{\mathfrak{g}_\alpha} = \alpha(H)^k(\mathrm{ad}(H+X)Y)_{\mathfrak{g}_\alpha} = \alpha(H)^{k+1}Y \neq 0$. This concludes the induction and shows that $H+X$ is not a nilpotent element in $\mathfrak{s}_\ell$. q.e.d.

The next result provides a necessary and sufficient condition in order that a Lie algebra isomorphism of $\mathfrak{n}$ can map a root space $\mathfrak{g}_\lambda$ onto another root space $\mathfrak{g}_\mu$. This result is of interest in its own right and will be applied later to settle the congruency problem.

Theorem 3.4. *Let $\lambda, \mu \in \Sigma^+$. Then there exists a Lie algebra isomorphism F of $\mathfrak{n}$ with $F(\mathfrak{g}_\lambda) = \mathfrak{g}_\mu$ if and only if there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\lambda) = \mu$.*

Proof. The if-part follows from standard theory of real semisimple Lie algebras. For the converse, let F be a Lie algebra isomorphism of $\mathfrak{n}$ with $F(\mathfrak{g}_\lambda) = \mathfrak{g}_\mu$. First of all we will show that without loss of generality we can assume that F preserves the gradation $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_m$. For $X \in \mathfrak{n}$ we denote by X_i the orthogonal projection of X onto $\mathfrak{n}_i$. We define a linear map $\widetilde{F} : \mathfrak{n} \to \mathfrak{n}\ ,\ X \mapsto \sum_{i=1}^m (F(X_i))_i$. Consider the lower central series of $\mathfrak{n}$, that is, $L_0(\mathfrak{n}) := \mathfrak{n}$, $L_1(\mathfrak{n}) := [\mathfrak{n},\mathfrak{n}]$, and $L_k(\mathfrak{n}) := [L_{k-1}(\mathfrak{n}),\mathfrak{n}] = \mathfrak{n}_{k+1} \oplus \cdots \oplus \mathfrak{n}_m$ $(2 \leq k \leq m-1)$. Since F is a Lie algebra isomorphism of $\mathfrak{n}$ it induces a vector space isomorphism of $L_k(\mathfrak{n})$ for all $k \in \{0 \ldots, m-1\}$. From this fact it follows easily that $\widetilde{F}$ is a vector space isomorphism of $\mathfrak{n}$. For $X_i \in \mathfrak{n}_i$ and $Y_j \in \mathfrak{n}_j$, $1 \leq i,j \leq m$, we have

$$\begin{aligned}
\widetilde{F}[X_i, Y_j] &= (F[X_i, Y_j])_{i+j} = [FX_i, FY_j]_{i+j} \\
&= [(FX_i)_i + \cdots + (FX_i)_m, (FY_j)_j + \cdots + (FY_j)_m]_{i+j} \\
&= [(FX_i)_i, (FY_j)_j]_{i+j} = [(FX_i)_i, (FY_j)_j] = [\widetilde{F}X_i, \widetilde{F}X_j]\ ,
\end{aligned}$$

which implies that $\widetilde{F}$ is a Lie algebra isomorphism of $\mathfrak{n}$ with $\widetilde{F}(\mathfrak{g}_\lambda) = \mathfrak{g}_\mu$. Thus we can assume from now on that F preserves the gradation $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_m$, that is, $F(\mathfrak{n}_i) = \mathfrak{n}_i$ for all $i \in \{1, \ldots, m\}$.

Since F preserves the gradation of $\mathfrak{n}$ it is clear that the level m_λ of λ must be equal to the level m_μ of μ. For a linear subspace V of $\mathfrak{n}$ we denote by $C(V;\mathfrak{n}_k)$ the centralizer of V in $\mathfrak{n}_k$. Obviously, when F preserves V then F preserves also $C(V;\mathfrak{n}_k)$ and the subalgebra of

$\mathfrak{n}$ generated by $C(V;\mathfrak{n}_k)$. Of particular importance for us will be the subalgebra $\mathfrak{n}(\widetilde{\alpha})$ generated by $C(\mathfrak{n}_{m-1};\mathfrak{n}_1)$. Note that $\mathfrak{n}(\widetilde{\alpha})$ is the subalgebra of $\mathfrak{n}$ generated by the root spaces of those simple roots in Λ that are not connected with the maximal root $\widetilde{\alpha}$ in the extended Dynkin diagram of Σ. Since F preserves $\mathfrak{n}(\widetilde{\alpha})$ we have either $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n}(\widetilde{\alpha})$ or $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n} \ominus \mathfrak{n}(\widetilde{\alpha})$. We now prove the theorem by considering the root systems case-by-case.

$\Sigma = A_r$, $r \geq 1$: For $r \in \{1,2\}$ this is trivial. Assume that $r \geq 3$. The subalgebra $\mathfrak{n}(\widetilde{\alpha})$ is of type A_{r-2} and generated by $\mathfrak{g}_{\alpha_2} \oplus \cdots \oplus \mathfrak{g}_{\alpha_{r-1}}$. Here, and in the following, when we say this subalgebra is of type A_{r-2} we refer to the nilpotent part in the induced Iwasawa decomposition of the semisimple subalgebra of $\mathfrak{g}$ of type A_{r-2} determined by the simple roots $\alpha_2, \ldots, \alpha_{r-2}$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n}(\widetilde{\alpha})$, we can apply an inductive argument, because every symmetry of the Dynkin diagram of the A_{r-2}-type subalgebra can be extended to a symmetry of the Dynkin diagram of A_r. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n} \ominus \mathfrak{n}(\widetilde{\alpha})$, we have $\{\lambda, \mu\} \in \{\alpha_1 + \cdots + \alpha_{m_\lambda}, \alpha_{r-m_\lambda+1} + \cdots + \alpha_r\}$, which implies that there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\lambda) = \mu$.

$\Sigma = B_r$, $r \geq 2$: For $r = 2$ this is trivial. In the case $r = 3$ the subalgebra $\mathfrak{n}(\widetilde{\alpha})$ is of type $A_1 \oplus A_1$ and equal to $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_3}$. This implies that F cannot map $\mathfrak{g}_{\alpha_2}$ onto $\mathfrak{g}_{\alpha_1}$ or $\mathfrak{g}_{\alpha_3}$, and vice versa. The subspace $[\mathfrak{g}_{\alpha_1}, [\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{\alpha_2}]]$ is trivial, whereas the subspace $[\mathfrak{g}_{\alpha_3}, [\mathfrak{g}_{\alpha_3}, \mathfrak{g}_{\alpha_2}]] \subset \mathfrak{g}_{\alpha_2+2\alpha_3}$ is nontrivial, and since the projection of $F(\mathfrak{g}_{\alpha_2})$ onto $\mathfrak{g}_{\alpha_2}$ must be equal to $\mathfrak{g}_{\alpha_2}$, this implies that F cannot map $\mathfrak{g}_{\alpha_1}$ onto $\mathfrak{g}_{\alpha_3}$, and vice versa. Since $C(\mathfrak{g}_{\alpha_1+\alpha_2};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}$ and $C(\mathfrak{g}_{\alpha_2+\alpha_3};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_2}$, the isomorphism F cannot map one of the two root spaces on level 2 onto the other one. Finally, $C(\mathfrak{g}_{\alpha_1+\alpha_2+\alpha_3};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}$ generates a subalgebra of type A_2, whereas $C(\mathfrak{g}_{\alpha_2+2\alpha_3};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_3}$ generates a subalgebra of type B_2, which implies that F cannot map one of the two root spaces on level 3 onto the other one. This proves the assertion for B_3. Now let $r \geq 4$. The subalgebra $\mathfrak{n}(\widetilde{\alpha})$ is the subalgebra of type $A_1 \oplus B_{r-2}$ generated by $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_3} \oplus \cdots \oplus \mathfrak{g}_{\alpha_r}$. This implies $F(\mathfrak{g}_{\alpha_1}) = \mathfrak{g}_{\alpha_1}$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n}(\widetilde{\alpha})$, we can apply an inductive argument to see that $\lambda = \mu$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \not\subset \mathfrak{n}(\widetilde{\alpha})$, then both λ and μ have an α_2-component. On each level $m_\lambda \in \{2, \ldots, 2r-2\}$ there are exactly two roots with an α_2-component, namely $\{\alpha_1 + \cdots + \alpha_{m_\lambda}, \alpha_2 + \cdots + \alpha_{m_\lambda+1}\}$ if $m_\lambda < r$, $\{\alpha_1 + \cdots + \alpha_r, \alpha_2 + \cdots + \alpha_{r-1} + 2\alpha_r\}$ if $m_\lambda = r$, and $\{\alpha_1 + \cdots + \alpha_{2r-m_\lambda} + 2\alpha_{2r-m_\lambda+1} + \cdots + 2\alpha_r, \alpha_2 + \cdots + \alpha_{2r-m_\lambda-1} + 2\alpha_{2r-m_\lambda} + \cdots + 2\alpha_r\}$ if $m_\lambda > r$. In all cases the centralizer of the first root in $\mathfrak{n}_1$ contains the invariant root space $\mathfrak{g}_{\alpha_1}$, whereas the centralizer of the second root in

$\mathfrak{n}_1$ does not contain this invariant subspace. This implies the assertion, since on the levels 1 and $2r-1=m$ there is only one root with an α_2-component.

$\Sigma = C_r$, $r \geq 2$: For $r=2$ this is the same as for B_2. For $r \geq 3$ we see that $\mathfrak{n}(\widetilde{\alpha})$ is a subalgebra of type C_{r-1} generated by $\mathfrak{g}_{\alpha_2} \oplus \dots \oplus \mathfrak{g}_{\alpha_r}$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n}(\widetilde{\alpha})$, we can apply an inductive argument to show that $\lambda = \mu$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n} \ominus \mathfrak{n}(\widetilde{\alpha})$, then both λ and μ must have an α_1-component. Since on each level there exists only one root with an α_1-component this implies $\lambda = \mu$.

$\Sigma = D_r$, $r \geq 3$: For $r=3$ this is the same as for A_3. If $r=4$, then $\mathfrak{n}(\widetilde{\alpha})$ is the subalgebra of type $A_1 \oplus A_1 \oplus A_1$ given by the root spaces $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4}$. Thus the root space $\mathfrak{g}_{\alpha_2}$ cannot be mapped onto any other simple root space under F, and vice versa. On the other hand, the symmetry group of the Dynkin diagram of D_4 is the symmetric group $\mathfrak{S}_3$ of three letters, and any two simple roots different from α_2 can be mapped to each other by such a symmetry. There are three roots on level 2 and three roots on level 3, and on each level any two of them can be mapped to each other by a symmetry in $\mathfrak{S}_3$. On higher levels there is only one root. Now assume that $r \geq 5$. In this case $\mathfrak{n}(\widetilde{\alpha})$ is a subalgebra of type $A_1 \oplus D_{r-2}$ generated by $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_3} \oplus \dots \oplus \mathfrak{g}_{\alpha_r}$. In particular, this implies $F(\mathfrak{g}_{\alpha_1}) = \mathfrak{g}_{\alpha_1}$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n}(\widetilde{\alpha})$, we again want to apply an inductive argument. It is clear that every symmetry of the Dynkin diagram of D_{r-2} can be extended to a symmetry of the Dynkin diagram of D_r if $r \neq 6$. We thus have to investigate the special case $r=6$. We have $C(\mathfrak{n}_7;\mathfrak{n}_1) = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_5} \oplus \mathfrak{g}_{\alpha_6}$, which generates a subalgebra of type $A_1 \oplus A_3$ and implies that $F(\mathfrak{g}_{\alpha_2}) = \mathfrak{g}_{\alpha_2}$. Next, we have $C(\mathfrak{n}_6;\mathfrak{n}_1) = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_5} \oplus \mathfrak{g}_{\alpha_6}$, which is a subalgebra of type $A_1 \oplus A_1 \oplus A_1$. Since F preserves $\mathfrak{g}_{\alpha_2}$, this implies $F(\mathfrak{g}_{\alpha_5} \oplus \mathfrak{g}_{\alpha_6}) = \mathfrak{g}_{\alpha_5} \oplus \mathfrak{g}_{\alpha_6}$, which shows that $r=6$ does not cause any problems for the inductive argument. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \not\subset \mathfrak{n}(\widetilde{\alpha})$, then both λ and μ have an α_2-component. On each level $m_\lambda \in \{2, \dots, 2r-5\}$ there are exactly two roots with an α_2-component modulo the nontrivial Dynkin diagram symmetry, namely $\{\alpha_1 + \dots + \alpha_{m_\lambda}, \alpha_2 + \dots + \alpha_{m_\lambda+1}\}$ if $m_\lambda < r-1$, $\{\alpha_1 + \dots + \alpha_r, \alpha_2 + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r\}$ if $m_\lambda = r$, and $\{\alpha_1 + \dots + \alpha_{2r-m_\lambda-2} + 2\alpha_{2r-m_\lambda-1} + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r, \alpha_2 + \dots + \alpha_{2r-m_\lambda-3} + 2\alpha_{2r-m_\lambda-2} + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r\}$ if $m_\lambda > r$. In all cases the centralizer of the second root in $\mathfrak{n}_1$ contains the invariant root space $\mathfrak{g}_{\alpha_1}$, whereas the centralizer of the first root in $\mathfrak{n}_1$ does not contain this invariant subspace. This implies the assertion, since on the levels 1, $2r-4$ and $2r-3=m$ there is only one root with an α_2-component.

$\Sigma = BC_r$, $r \geq 2$: For $r = 2$ we have $\mathfrak{n}(\widetilde{\alpha}) = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{2\alpha_2}$, which readily implies that F cannot map any root space onto another root space. For $r \geq 3$, $\mathfrak{n}(\widetilde{\alpha})$ is a subalgebra of type BC_{r-1} generated by $\mathfrak{g}_{\alpha_2} \oplus \cdots \oplus \mathfrak{g}_{\alpha_r}$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n}(\widetilde{\alpha})$, we can apply an inductive argument to show that $\lambda = \mu$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n} \ominus \mathfrak{n}(\widetilde{\alpha})$, then both λ and μ must have an α_1-component. Since on each level $m_\lambda \in \{1, \ldots, 2r\}$ there exists exactly one root with an α_1-component this implies $\lambda = \mu$.

$\Sigma = E_6$: The root $\beta_{10} = \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ is the only root on level 10 and $C(\mathfrak{g}_{\beta_{10}}; \mathfrak{n}_1) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_3} \oplus \cdots \oplus \mathfrak{g}_{\alpha_6}$, which generates a subalgebra of type A_5. This shows that F leaves the subalgebra generated by $\mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4} \oplus \mathfrak{g}_{\alpha_5}$ invariant, which is a subalgebra of type A_3. Using $\mathfrak{n}(\widetilde{\alpha})$ for this A_3-type subalgebra implies $F(\mathfrak{g}_{\alpha_4}) = \mathfrak{g}_{\alpha_4}$. The root $\beta_9 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$ is the only root on level 9 and $C(\mathfrak{g}_{\beta_9}; \mathfrak{n}_1) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_5} \oplus \mathfrak{g}_{\alpha_6}$, which generates a subalgebra of type $A_2 \oplus A_1 \oplus A_2$. The A_1-part gives $F(\mathfrak{g}_{\alpha_2}) = \mathfrak{g}_{\alpha_2}$. Taking the intersection of the $(A_2 \oplus A_1 \oplus A_2)$-type subalgebra with the A_3-type subalgebra obtained in level 10 we see that $F(\mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_5}) = \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_5}$. The subalgebra $\mathfrak{n}(\widetilde{\alpha})$ is of type A_5 and generated by $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_3} \oplus \cdots \oplus \mathfrak{g}_{\alpha_6}$. If $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n}(\widetilde{\alpha})$, there exists a transformation in the symmetry group of the Dynkin diagram of A_5 mapping λ to μ. The structure of the Dynkin diagram of E_6 shows that this symmetry extends to a symmetry of the Dynkin diagram of E_6. Now assume that $\mathfrak{g}_\lambda, \mathfrak{g}_\mu \subset \mathfrak{n} \ominus \mathfrak{n}(\widetilde{\alpha})$. From the structure of the root system E_6, and taking into account the symmetry group of the Dynkin diagram of E_6, it remains to show that the root spaces corresponding to the following roots cannot be mapped to each other under F:

on level 4: $\beta_4^1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4,\ \beta_4^2 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$;

on level 5: $\beta_5^1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,\ \beta_5^2 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$;

on level 6: $\beta_6^1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,\ \beta_6^2 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$;

on level 7: $\beta_7^1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6,\ \beta_7^2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$.

These are exactly all pairs of roots in Σ^+ modulo $\mathrm{Aut}(DD)$-congruence with an α_1-component. The assertion then follows by considering the following centralizers:

on level 4: $\mathfrak{g}_{\beta_4^1} \subset C(\mathfrak{g}_{\alpha_4}; \mathfrak{n}_4),\ \mathfrak{g}_{\beta_4^2} \not\subset C(\mathfrak{g}_{\alpha_4}; \mathfrak{n}_4)$;

on level 5: $\mathfrak{g}_{\beta_5^1} \subset C(\mathfrak{g}_{\alpha_2}; \mathfrak{n}_5),\ \mathfrak{g}_{\beta_5^2} \not\subset C(\mathfrak{g}_{\alpha_2}; \mathfrak{n}_5)$;

on level 6: $\mathfrak{g}_{\beta_6^1} \not\subset C(\mathfrak{g}_{\alpha_4}; \mathfrak{n}_6),\ \mathfrak{g}_{\beta_6^2} \subset C(\mathfrak{g}_{\alpha_4}; \mathfrak{n}_6)$;

on level 7: $\mathfrak{g}_{\beta_7^1} \not\subset C(\mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_5}; \mathfrak{n}_7),\ \mathfrak{g}_{\beta_7^2} \subset C(\mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_5}; \mathfrak{n}_7)$, and the fact that all these centralizers are invariant under F.

$\Sigma = E_7$: The root $\beta_{16} = \alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$ is the only root on level 16 and $C(\mathfrak{g}_{\beta_{16}};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_2}\oplus\cdots\oplus\mathfrak{g}_{\alpha_7}$, which generates a subalgebra of type D_6. Applying the $\mathfrak{n}(\widetilde{\alpha})$-method to this subalgebra shows that F leaves the subalgebra generated by $\mathfrak{g}_{\alpha_2}\oplus\cdots\oplus\mathfrak{g}_{\alpha_5}\oplus\mathfrak{g}_{\alpha_7}$ invariant, which is a subalgebra of type $D_4\oplus A_1$. This implies $F(\mathfrak{g}_{\alpha_7}) = \mathfrak{g}_{\alpha_7}$. The root $\beta_{15} = \alpha_1+2\alpha_2+2\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$ is the only root on level 15 and $C(\mathfrak{g}_{\beta_{15}};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_1}\oplus\mathfrak{g}_{\alpha_2}\oplus\mathfrak{g}_{\alpha_4}\oplus\cdots\oplus\mathfrak{g}_{\alpha_7}$, which generates a subalgebra of type $A_1\oplus A_5$. This implies $F(\mathfrak{g}_{\alpha_1}) = \mathfrak{g}_{\alpha_1}$. Moreover, applying the $\mathfrak{n}(\widetilde{\alpha})$-method to the subalgebra of type A_5 implies that F leaves the subalgebra generated by $\mathfrak{g}_{\alpha_4}\oplus\mathfrak{g}_{\alpha_5}\oplus\mathfrak{g}_{\alpha_6}$ invariant, which is a subalgebra of type A_3. Applying once again the $\mathfrak{n}(\widetilde{\alpha})$-method to this A_3-type subalgebra shows that $F(\mathfrak{g}_{\alpha_5}) = \mathfrak{g}_{\alpha_5}$. The root $\beta_{14} = \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$ is the only root on level 14 and $C(\mathfrak{g}_{\beta_{14}};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_1}\oplus\mathfrak{g}_{\alpha_2}\oplus\mathfrak{g}_{\alpha_3}\oplus\mathfrak{g}_{\alpha_5}\oplus\mathfrak{g}_{\alpha_6}\oplus\mathfrak{g}_{\alpha_7}$, which generates a subalgebra of type $A_2\oplus A_1\oplus A_3$. The A_1-part implies $F(\mathfrak{g}_{\alpha_2}) = \mathfrak{g}_{\alpha_2}$, and applying the $\mathfrak{n}(\widetilde{\alpha})$-method to the A_3-part implies $F(\mathfrak{g}_{\alpha_6}) = \mathfrak{g}_{\alpha_6}$. Taking the intersection of the D_6-type subalgebra obtained in level 16 and the A_2-type subalgebra obtained in level 14 implies $F(\mathfrak{g}_{\alpha_3}) = \mathfrak{g}_{\alpha_3}$. On level 13 there are two roots, namely $\beta_{13}^1 = \alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+2\alpha_6+\alpha_7$ and $\beta_{13}^2 = \alpha_1+\alpha_2+2\alpha_3+3\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$, and $C(\mathfrak{g}_{\beta_{13}^1}\oplus\mathfrak{g}_{\beta_{13}^2};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_1}\oplus\mathfrak{g}_{\alpha_3}\oplus\mathfrak{g}_{\alpha_4}\oplus\mathfrak{g}_{\alpha_6}\oplus\mathfrak{g}_{\alpha_7}$, which generates a subalgebra of type $A_3\oplus A_2$. Taking the intersection of the A_3-type subalgebra with the A_5-type subalgebra obtained in level 15 implies $F(\mathfrak{g}_{\alpha_4}) = \mathfrak{g}_{\alpha_4}$. Hence all simple roots spaces are preserved by F, which implies $F(\mathfrak{g}_\lambda) = \mathfrak{g}_\lambda$ for all $\lambda\in\Sigma^+$.

$\Sigma = E_8$: The root $\beta_{28} = 2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+\alpha_8$ is the only root on level 28 and $C(\mathfrak{g}_{\beta_{28}};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_1}\oplus\cdots\oplus\mathfrak{g}_{\alpha_7}$, which generates a subalgebra of type E_7. Since F preserves this subalgebra, the discussion in the previous case shows that $F(\mathfrak{g}_{\alpha_i}) = \mathfrak{g}_{\alpha_i}$ for all $i\in\{1,\ldots,7\}$. The root $\beta_{27} = 2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+2\alpha_7+\alpha_8$ is the only root on level 27 and $C(\mathfrak{g}_{\beta_{27}};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_1}\oplus\cdots\oplus\mathfrak{g}_{\alpha_6}\oplus\mathfrak{g}_{\alpha_8}$, which generates a subalgebra of type $E_6\oplus A_1$. This shows $F(\mathfrak{g}_{\alpha_8}) = \mathfrak{g}_{\alpha_8}$. Hence F preserves all simple root spaces, which implies $F(\mathfrak{g}_\lambda) = \mathfrak{g}_\lambda$ for all $\lambda\in\Sigma^+$.

$\Sigma = F_4$: The root $\beta_{10} = \alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$ is the only root on level 10 and $C(\mathfrak{g}_{\beta_{10}};\mathfrak{n}_1) = \mathfrak{g}_{\alpha_2}\oplus\mathfrak{g}_{\alpha_3}\oplus\mathfrak{g}_{\alpha_4}$, which generates a subalgebra of type C_3. This subalgebra is invariant under F, and applying the $\mathfrak{n}(\widetilde{\alpha})$-method to it shows that the subalgebra generated by $\mathfrak{g}_{\alpha_2}\oplus\mathfrak{g}_{\alpha_3}$ is invariant under F, which is a subalgebra of type B_2. Applying the

$\mathfrak{n}(\widetilde{\alpha})$-method to this B_2-type subalgebra implies that $F(\mathfrak{g}_{\alpha_2}) = \mathfrak{g}_{\alpha_2}$. The root $\beta_9 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4$ is the only root on level 9 and $C(\mathfrak{g}_{\beta_9}; \mathfrak{n}_1) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4}$, which generates a subalgebra of type $A_1 \oplus A_2$. This implies $F(\mathfrak{g}_{\alpha_1}) = \mathfrak{g}_{\alpha_1}$ and $F(\mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4}) = \mathfrak{g}_{\alpha_3} \oplus \mathfrak{g}_{\alpha_4}$. Since F leaves also $\mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_3}$ invariant, the latter statement implies $F(\mathfrak{g}_{\alpha_3}) = \mathfrak{g}_{\alpha_3}$. Finally, the root $\beta_8 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ is the only root on level 8 and $C(\mathfrak{g}_{\beta_8}; \mathfrak{n}_1) = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_4}$, which generates a subalgebra of type $A_2 \oplus A_1$. This implies $F(\mathfrak{g}_{\alpha_4}) = \mathfrak{g}_{\alpha_4}$ and thus all simple root spaces are preserved by F, which implies $F(\mathfrak{g}_\lambda) = \mathfrak{g}_\lambda$ for all $\lambda \in \Sigma^+$.

$\Sigma = G_2$: The root $\beta_4 = 3\alpha_1 + \alpha_2$ is the only root on level 4 and $C(\mathfrak{g}_{\beta_4}; \mathfrak{n}_1) = \mathfrak{g}_{\alpha_1}$, which implies $F(\mathfrak{g}_{\alpha_1}) = \mathfrak{g}_{\alpha_1}$. The root $\beta_3 = 2\alpha_1 + \alpha_2$ is the only root on level 3 and $C(\mathfrak{g}_{\beta_3}; \mathfrak{n}_1) = \mathfrak{g}_{\alpha_2}$, which implies $F(\mathfrak{g}_{\alpha_2}) = \mathfrak{g}_{\alpha_2}$. Altogether this implies $F(\mathfrak{g}_\lambda) = \mathfrak{g}_\lambda$ for all $\lambda \in \Sigma^+$. q.e.d.

We now start to investigate the congruency problem for the foliations $\mathfrak{F}_\ell$. We assume $r \geq 2$ from now on. According to Proposition 3.1 all leaves of $\mathfrak{F}_\ell$ and all leaves of $\mathfrak{F}_{\ell'}$ are isometrically congruent to each other. Therefore the two leaves $S_\ell \cdot o$ and $S_{\ell'} \cdot o$ are isometrically congruent to each other. Since S_ℓ is connected, it follows from Lemma 3.3 that S_ℓ is completely solvable, that is, there exists a basis of $\mathfrak{s}_\ell$ such that $\mathrm{Ad}(S_\ell)$ consists of upper triangular matrices with respect to that basis. Since any two connected completely solvable transitive groups of isometries on a Riemannian manifold are conjugate in the isometry group of that manifold (see [1]), we conclude that the Lie algebras $\mathfrak{s}_\ell$ and $\mathfrak{s}_{\ell'}$ are isomorphic.

Let $F : \mathfrak{s}_\ell \to \mathfrak{s}_{\ell'}$ be a Lie algebra isomorphism. Then F maps the Cartan subalgebra $\mathfrak{c}_\ell$ of $\mathfrak{s}_\ell$ onto a Cartan subalgebra of $\mathfrak{s}_{\ell'}$. Since any two Cartan subalgebras of a solvable Lie algebra are conjugate to each other under an inner automorphism we can assume that $F(\mathfrak{c}_\ell) = \mathfrak{c}_{\ell'}$. In view of Lemma 3.3 it is natural to distinguish the following cases:

Case 1. $\ell = \mathbb{R}H_\lambda$ for some root $\lambda \in \Sigma^+$ with $2\lambda \in \Sigma^+$.

Case 2. $\ell = \mathbb{R}H_\lambda$ for some root $\lambda \in \Sigma^+$ with $2\lambda, \frac{1}{2}\lambda \notin \Sigma^+$.

Case 3. $\ell \neq \mathbb{R}H_\lambda$ for all $\lambda \in \Sigma^+$.

In the first case $\mathfrak{c}_\ell$ is a nilpotent and nonabelian Cartan subalgebra, in the second case $\mathfrak{c}_\ell$ is an abelian Cartan subalgebra containing nontrivial nilpotent elements, and in the third case $\mathfrak{c}_\ell$ is an abelian Cartan subalgebra without nontrivial nilpotent elements.

We start with Case 1. Assume that $\ell = \mathbb{R}H_\lambda$ for some root $\lambda \in \Sigma^+$ with $2\lambda \in \Sigma^+$. From Lemma 3.3 we see that the Cartan subalgebra $\mathfrak{c}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{g}_\lambda \oplus \mathfrak{g}_{2\lambda}$ of $\mathfrak{s}_\ell$ is nilpotent and nonabelian. Thus the Cartan subalgebra $\mathfrak{c}_{\ell'}$ of $\mathfrak{s}_{\ell'}$ must also be nilpotent and nonabelian, and we get from Lemma 3.3 that there exists a root $\lambda' \in \Sigma^+$ with $2\lambda' \in \Sigma^+$ such that $\ell' = \mathbb{R}H_{\lambda'}$. The derived subalgebra of $\mathfrak{c}_\ell$ is $\mathfrak{g}_{2\lambda}$ and must be mapped onto the derived subalgebra $\mathfrak{g}_{2\lambda'}$ of $\mathfrak{c}_{\ell'}$ by F. We can now apply Theorem 3.4 and get a symmetry $P \in \mathrm{Aut}(DD)$ with $P(2\lambda) = 2\lambda'$ and hence also $P(\lambda) = \lambda'$. By construction, this implies then $P(\ell) = \ell'$.

Next, we consider Case 2. Let $\ell = \mathbb{R}H_\lambda$ for some root $\lambda \in \Sigma^+$ with $2\lambda, \frac{1}{2}\lambda \notin \Sigma^+$. Then the Cartan subalgebra $\mathfrak{c}_\ell$ of $\mathfrak{s}_\ell$ is abelian and $\dim \mathfrak{c}_\ell = r - 1 + \dim \mathfrak{g}_\lambda$. Since F maps $\mathfrak{c}_\ell$ onto the Cartan subalgebra $\mathfrak{c}_{\ell'}$ of $\mathfrak{s}_{\ell'}$, the latter one must also be abelian and of dimension greater than $r - 1$. According to Lemma 3.3 this can happen only when $\ell' = \mathbb{R}H_{\lambda'}$ for some root $\lambda' \in \Sigma^+$ with $2\lambda', \frac{1}{2}\lambda' \notin \Sigma^+$. The isomorphism F maps the Cartan subalgebra $\mathfrak{c}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{g}_\lambda$ of $\mathfrak{s}_\ell$ to the Cartan subalgebra $\mathfrak{c}_{\ell'} = (\mathfrak{a} \ominus \ell') \oplus \mathfrak{g}_{\lambda'}$ of $\mathfrak{s}_{\ell'}$. Moreover, F maps the set of nilpotent elements in $\mathfrak{s}_\ell$ onto the set of nilpotent elements in $\mathfrak{s}_{\ell'}$, which for both subalgebras is equal to $\mathfrak{n}$. It follows that $\mathfrak{g}_\lambda = \mathfrak{c}_\ell \cap \mathfrak{n}$ is mapped onto $\mathfrak{g}_{\lambda'} = \mathfrak{c}_{\ell'} \cap \mathfrak{n}$ by F. From Theorem 3.4 we then conclude that there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\lambda) = \lambda'$ and hence also $P(\ell) = \ell'$.

We finally consider Case 3. Then $\mathfrak{c}_\ell = \mathfrak{a} \ominus \ell$ is a Cartan subalgebra of $\mathfrak{s}_\ell$, which is mapped by the isomorphism F to the Cartan subalgebra $\mathfrak{c}_{\ell'} = \mathfrak{a} \ominus \ell'$ of $\mathfrak{s}_{\ell'}$. Using Lemma 3.3 we therefore must have $\ell' \neq \mathbb{R}H_\lambda$ for all $\lambda \in \Sigma^+$. If $\lambda \in \Sigma^+$ then $\lambda|\mathfrak{c}_\ell$ is a root of $\mathfrak{s}_\ell$ with respect to the Cartan subalgebra $\mathfrak{c}_\ell$. We denote by Σ^+_ℓ the set of these roots and by $\mathfrak{g}^\ell_\lambda$ the corresponding root spaces. The set $\Lambda_\ell = \{\alpha|\mathfrak{c}_\ell \mid \alpha \in \Lambda\}$ generates Σ^+_ℓ in a similar way as Λ generates Σ^+. For each $\lambda \in \Sigma^+_\ell$ we define a linear map $F^*\lambda : \mathfrak{c}_{\ell'} \to \mathbb{R}$ by $F^*\lambda(H) = \lambda(F^{-1}H)$ for all $H \in \mathfrak{c}_{\ell'}$. Since $[H, FX] = F[F^{-1}H, X] = \lambda(F^{-1}H)FX$ for all $X \in \mathfrak{g}^\ell_\lambda$, we see that $F^*\lambda \in \Sigma^+_{\ell'}$ and $F(\mathfrak{g}^\ell_\lambda) = \mathfrak{g}^{\ell'}_{F^*\lambda}$ for all $\lambda \in \Sigma^+_\ell$. Thus F^* induces an isomorphism from the set Σ^+_ℓ onto the set $\Sigma^+_{\ell'}$ and maps root spaces of roots in Σ^+_ℓ onto root spaces of roots in $\Sigma^+_{\ell'}$. Since $F^*(\lambda+\mu) = F^*\lambda+F^*\mu$ for all $\lambda, \mu \in \Sigma^+_\ell$, F^* maps root strings to root strings, and hence induces an isomorphism between the sets Λ_ℓ and $\Lambda_{\ell'}$.

It may happen that there exist two roots $\lambda, \mu \in \Sigma^+$ such that $\lambda|\mathfrak{c}_\ell = \mu|\mathfrak{c}_\ell$. In this case $\mathfrak{g}_\lambda \oplus \mathfrak{g}_\mu$ is the root space $\mathfrak{g}^\ell_\lambda = \mathfrak{g}^\ell_\mu$ corresponding to $\lambda|\mathfrak{c}_\ell = \mu|\mathfrak{c}_\ell \in \Sigma^+_\ell$, and we say that the two roots λ and μ collapse in $\mathfrak{s}_\ell$. Note that if λ and μ collapse in $\mathfrak{s}_\ell$, and if $\kappa \in \Sigma^+$ such that

$\kappa + k\lambda, \kappa + k\mu \in \Sigma^+$ for some $k > 0$, then also $\kappa + k\lambda$ and $\kappa + k\mu$ collapse in $\mathfrak{s}_\ell$. If $\lambda \in \Sigma^+$ does not collapse in $\mathfrak{s}_\ell$ with any other root in Σ^+ then $\mathfrak{g}^\ell_\lambda = \mathfrak{g}_\lambda$.

If no two simple roots in Λ collapse in $\mathfrak{s}_\ell$, then Λ_ℓ consists of r elements. Since F^* maps the set Λ_ℓ isomorphically onto the set $\Lambda_{\ell'}$, the latter set consists also of r elements and it follows that no two simple roots in Λ collapse in $\mathfrak{s}_{\ell'}$. Since F^* maps root strings to root strings we conclude that F^* induces a Dynkin diagram symmetry P. Let k be an isometry in the normalizer of $\mathfrak{a} + \mathfrak{n}$ in K such that the action of $\mathrm{Ad}(k)^*$ on Λ coincides with the action of P. For all $\alpha \in \Lambda$, $X_\alpha \in \mathfrak{g}_\alpha$ and $H' \in \mathfrak{c}_{\ell'}$ we have $[H', FX_\alpha] = F[F^{-1}H', X_\alpha]$, as F is a Lie algebra isomorphism. The left-hand side of the previous equation is equal to $(F^*\alpha)(H')FX_\alpha = (\mathrm{Ad}(k)^*\alpha)(H')FX_\alpha = \alpha(\mathrm{Ad}(k)^{-1}H')FX_\alpha$, and the right-hand side equals $\alpha(F^{-1}H')FX_\alpha$. Comparing the two sides shows that $\alpha(\mathrm{Ad}(k)^{-1}H') = \alpha(F^{-1}H')$ for all $\alpha \in \Lambda$ and $H' \in \mathfrak{c}_{\ell'}$. This implies $\mathrm{Ad}(k)^{-1}|\mathfrak{c}_{\ell'} = F^{-1}|\mathfrak{c}_{\ell'}$ and hence $\mathrm{Ad}(k)|\mathfrak{c}_\ell = F|\mathfrak{c}_\ell$. Then we have $\mathrm{Ad}(k)(\mathfrak{s}_\ell) = \mathfrak{s}_{\ell'}$ and hence, since $\mathrm{Ad}(k)$ preserves the inner product on $\mathfrak{a}$, also $\mathrm{Ad}(k)(\ell) = \ell'$. Thus $P \in \mathrm{Aut}(DD)$ is a symmetry with $P(\ell) = \ell'$.

Now assume that two simple roots $\alpha, \beta \in \Lambda$ collapse in $\mathfrak{s}_\ell$. If $r = 2$, we must have $\ell = \mathbb{R}(H_{\alpha_1} - H_{\alpha_2}) = \ell'$. We therefore can assume $r \geq 3$ from now on. Then Λ_ℓ consists of $r - 1$ elements, and since $\Lambda_{\ell'}$ is isomorphic to Λ_ℓ, we see that there exist two simple roots $\alpha', \beta' \in \Lambda$ that collapse in $\mathfrak{s}_{\ell'}$. We claim that $F(\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta) = \mathfrak{g}_{\alpha'} \oplus \mathfrak{g}_{\beta'}$. Assume that this is not true. Then there exists a simple root $\gamma \in \Lambda \setminus \{\alpha', \beta'\}$ such that $F(\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta) = \mathfrak{g}_\gamma$. By inspection of the possible root systems and the multiplicities of the roots (see [2] for a list of the multiplicities), we see that this is possible only in the following cases:

(1) $\Sigma = B_r$ and $M = \mathrm{SO}^o(r+2, r)/\mathrm{SO}(r+2) \times \mathrm{SO}(r)$;

(2) $\Sigma = BC_r$ and $M = \mathrm{SU}(r+2, r)/\mathrm{S}(\mathrm{U}(r+2) \times \mathrm{U}(r))$;

(3) $\Sigma = BC_r$ and $M = \mathrm{Sp}(r+2, r)/\mathrm{Sp}(r+2) \times \mathrm{Sp}(r)$;

(4) $\Sigma = F_4$ and $M = E_6^2/\mathrm{SU}(6) \times \mathrm{SU}(2)$.

Since α and β are on the same level it follows that whenever two roots in Σ^+ collapse in $\mathfrak{s}_\ell$ they are on the same level in Σ^+. This implies that F preserves the gradation $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_m$ of $\mathfrak{n}$. In Cases (1), (2) and (3) we must have $\gamma = \alpha_r$. In Case (1) the centralizer of $\mathfrak{g}_{\alpha_r}$ in $\mathfrak{n}_1$ is $\mathfrak{n}_1 \ominus \mathfrak{g}_{\alpha_{r-1}}$, which is a linear subspace of codimension one in $\mathfrak{n}_1$. On the other hand, the orthogonal complement of the centralizer of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ in

$\mathfrak{n}_1$ contains at least two simple root spaces and hence has a codimension greater than one in $\mathfrak{n}_1$. This excludes Case (1). In Cases (2) and (3) we consider derived subalgebras. Since F maps $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ isomorphically onto $\mathfrak{g}_{\alpha_r}$, it also maps their derived subalgebras isomorphically onto each other. But $[\mathfrak{g}_{\alpha_r}, \mathfrak{g}_{\alpha_r}] = \mathfrak{g}_{2\alpha_r}$ has odd dimension, whereas $[\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta, \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta] = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ has even dimension, which gives a contradiction. Finally, in Case (4), we must have $\gamma \in \{\alpha_3, \alpha_4\}$. We consider again centralizers to derive a contradiction. The centralizer of $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ in $\mathfrak{n}_1$ is $\mathfrak{g}_{\alpha_4}$ and has dimension 2. On the other hand, the centralizer of $\mathfrak{g}_{\alpha_3}$ in $\mathfrak{n}_1$ is $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_3}$, which is 3-dimensional, and the centralizer of $\mathfrak{g}_{\alpha_4}$ in $\mathfrak{n}_1$ is $\mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_4}$, which is 4-dimensional. This shows that F cannot map $\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$ isomorphically onto $\mathfrak{g}_\gamma$. We thus have proved that $F(\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta) = \mathfrak{g}_{\alpha'} \oplus \mathfrak{g}_{\beta'}$.

Since $r \geq 3$, the sets $\{\gamma \in \Sigma^+ \mid \langle \gamma, \alpha \rangle = 0\}$ and $\{\gamma \in \Sigma^+ \mid \langle \gamma, \beta \rangle = 0\}$ are nonempty and not identical. Thus there exists a root $\gamma \in \Sigma^+$ with $\langle \gamma, \alpha \rangle = 0$ and $\langle \gamma, \beta \rangle \neq 0$. This means that $\gamma \pm \alpha \notin \Sigma^+$ and $\gamma + \beta \in \Sigma^+$. Assume that γ collapses in $\mathfrak{s}_\ell$ with some root $\mu \in \Sigma^+$. Then there exist a root $\delta \in \Sigma^+$ and a positive integer k such that $\gamma = \delta + k\alpha$ and $\mu = \delta + k\beta$. Then $\gamma - k\alpha = \delta \in \Sigma^+$, which implies $\gamma - \alpha \in \Sigma^+$ by general properties of root strings. This contradicts $\gamma - \alpha \notin \Sigma^+$, and we conclude that γ does not collapse in $\mathfrak{s}_\ell$ with any root in Σ^+. We can characterize $\mathfrak{g}_\alpha$ by $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \mid [X, \mathfrak{g}_\gamma] = 0\}$. Then $F(\mathfrak{g}_\alpha) = \{Y \in \mathfrak{g}_{\alpha'} \oplus \mathfrak{g}_{\beta'} \mid Y = FX \text{ and } [X, \mathfrak{g}_\gamma] = 0 \text{ for some } X \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta\}$. Since γ does not collapse in $\mathfrak{s}_\ell$, there exists a root $\gamma' \in \Sigma^+$ such that $F^*\gamma|\mathfrak{c}_{\ell'} = \gamma'|\mathfrak{c}_{\ell'}$, and we get $F(\mathfrak{g}_\gamma) = \mathfrak{g}_{\gamma'}$ and hence $F(\mathfrak{g}_\alpha) = \{Y \in \mathfrak{g}_{\alpha'} \oplus \mathfrak{g}_{\beta'} \mid [Y, \mathfrak{g}_{\gamma'}] = 0\}$. If $\alpha' + \gamma', \beta' + \gamma' \in \Sigma^+$, then $F(\mathfrak{g}_\alpha) = 0$, which contradicts the fact that F is an isomorphism. If $\alpha' + \gamma', \beta' + \gamma' \notin \Sigma^+$ then $F(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha'} \oplus \mathfrak{g}_{\beta'}$, which is again a contradiction. We therefore must have either $\alpha' + \gamma' \in \Sigma^+, \beta' + \gamma' \notin \Sigma^+$, which implies $F(\mathfrak{g}_\alpha) = \mathfrak{g}_{\beta'}$, or $\alpha' + \gamma' \notin \Sigma^+, \beta' + \gamma' \in \Sigma^+$, which implies $F(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha'}$. We therefore can assume, without loss of generality, that $F(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha'}$ and $F(\mathfrak{g}_\beta) = \mathfrak{g}_{\beta'}$. From Theorem 3.4 we conclude that there exist $P, Q \in \mathrm{Aut}(DD)$ with $P(\alpha) = \alpha'$ and $Q(\beta) = \beta'$.

We claim that we can choose $P = Q$. If α or β is in the set $\mathrm{Fix}(\Lambda, \mathrm{Aut}(DD))$ of fixed points of the action of $\mathrm{Aut}(DD)$ on Λ, then this is clear. Now assume that $\alpha, \beta \notin \mathrm{Fix}(\Lambda, \mathrm{Aut}(DD))$. Then we have necessarily $\Sigma \in \{A_r, D_r, E_6\}$. If $\Sigma = D_4$, we have $\mathrm{Aut}(DD) = \mathfrak{S}_3$, and for any two pairs of distinct roots not in $\mathrm{Fix}(\Lambda, \mathrm{Aut}(DD))$ there exists a symmetry in $\mathfrak{S}_3$ mapping the first pair to the second one. This shows that we can choose $P = Q$ if $\Sigma = D_4$. If $\Sigma = D_r$, $r \neq 4$, the set

$\Lambda \setminus \mathrm{Fix}(\Lambda, \mathrm{Aut}(DD))$ consists of exactly two points, which readily implies that we can choose $P = Q$. If $\Sigma = E_6$, the structure of the Dynkin diagram of E_6 shows that $\Lambda \setminus \mathrm{Fix}(\Lambda, \mathrm{Aut}(DD))$ consists of two pairs of simple roots, where in each pair the roots form nontrivial strings with each other, but no nontrivial strings with the roots of the other pair. If $\{\alpha, \beta\}$ forms such a pair, the other pair consists of simple roots of Λ that do not collapse in $\mathfrak{s}_\ell$ and hence correspond to roots in Λ_ℓ. Since F preserves strings of roots in Λ_ℓ, we conclude that $\{\alpha', \beta'\}$ must also be one of the two pairs. So the structure of the Dynkin diagram of E_6 tells us that we must have $P = Q$. If α and β are in different pairs, a similar argument gives $P = Q$ as well. Finally, consider the case $\Sigma = A_r$, in which case we have $\mathrm{Aut}(DD) = \mathbb{Z}_2$. If $P \neq Q$, then either P or Q must be the identity, say Q. Then $\beta = \beta'$, and just contemplating about the structure of the Dynkin diagram of A_r and taking into account that string relations between roots in $\Lambda \setminus \{\alpha, \beta\}$ are preserved by F, shows that also P must be the identity, which is a contradiction. Thus either both P and Q are the identity, or both of them are different from the identity in $\mathrm{Aut}(DD)$. Altogether this implies that we can choose $P = Q$. Since $\ell = \mathbb{R}(H_\alpha - H_\beta)$ and $\ell' = \mathbb{R}(H_{\alpha'} - H_{\beta'})$, we can now conclude that there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\ell) = \ell'$.

Conversely, let ℓ, ℓ' be two different lines and assume that there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\ell) = \ell'$. There exists an outer automorphism F of $\mathfrak{g}$ such that the induced action of F on Σ^+ is equal to the action of P on Σ^+. In fact, $F = \mathrm{Ad}(k)$ for some k in the normalizer of $\mathfrak{a} + \mathfrak{n}$ in K. By construction we have $F(\mathfrak{s}_\ell) = \mathfrak{s}_{\ell'}$, which implies that k is an isometry of M mapping the leaves of the foliation $\mathfrak{F}_\ell$ onto the leaves of the foliation $\mathfrak{F}_{\ell'}$. This finishes the investigations about the congruency of the foliations $\mathfrak{F}_\ell$, and we summarize it in:

Theorem 3.5. *Two foliations $\mathfrak{F}_\ell$ and $\mathfrak{F}_{\ell'}$ are isometrically congruent to each other if and only if there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\ell) = \ell'$.*

4. The foliations $\mathfrak{F}_i$

The following lemma is crucial for showing that the foliations $\mathfrak{F}_i$ are well-defined:

Lemma 4.1. *Let $\alpha \in \Lambda$ be a simple root. For each unit vector $\xi \in \mathfrak{g}_\alpha$ the subspace $\mathfrak{s}_\xi = \mathfrak{a} + (\mathfrak{n} \ominus \mathbb{R}\xi)$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$. Moreover, if ξ, η are two unit vectors in $\mathfrak{g}_\alpha$, then there exists an isometry k in the centralizer of $\mathfrak{a}$ in K^o such that $\mathrm{Ad}(k)(\mathfrak{s}_\xi) = \mathfrak{s}_\eta$.*

Proof. The fact that $\mathfrak{s}_\xi$ is a subalgebra of $\mathfrak{a}+\mathfrak{n}$ follows immediately from elementary properties of root systems. For the congruency problem there is nothing to prove if $\dim \mathfrak{g}_\alpha = 1$, and for $\dim \mathfrak{g}_\alpha > 1$ this follows from Exercise 2 on p. 211 in [11] (see p. 566 for the solution). q.e.d.

The previous lemma implies that for each simple root $\alpha_i \in \Lambda$ we obtain a congruence class of homogeneous foliations of codimension one on M. More precisely, let $\alpha_i \in \Lambda$, $\xi \in \mathfrak{g}_{\alpha_i}$ be a unit vector, and S_ξ be the connected Lie subgroup of AN with Lie algebra $\mathfrak{s}_\xi$. Then the orbits of the action of S_ξ on M form a homogeneous foliation $\mathfrak{F}_\xi$ of codimension one on M. If $\eta \in \mathfrak{g}_{\alpha_i}$ is another unit vector, it follows from Lemma 4.1 that the induced foliation $\mathfrak{F}_\eta$ is congruent to $\mathfrak{F}_\xi$ under an isometry in the centralizer of $\mathfrak{a}$ in K^o. We denote by $\mathfrak{F}_i$ a representative of this congruence class of homogeneous foliations of codimension one on M, that is, $\mathfrak{F}_i = \mathfrak{F}_\xi$ for some unit vector $\xi \in \mathfrak{g}_{\alpha_i}$.

Our next aim is to study the geometry of such a foliation $\mathfrak{F}_i$ in more detail. Let $\alpha_i \in \Lambda$ be a simple root and $\xi \in \mathfrak{g}_{\alpha_i}$ be a unit vector such that $\mathfrak{F}_i = \mathfrak{F}_\xi$. We put $H_i = H_{\alpha_i}/|H_{\alpha_i}| = H_{\alpha_i}/|\alpha_i|$. The vectors ξ and H_i span a two-dimensional subalgebra of $\mathfrak{a}+\mathfrak{n}$ with Lie bracket $[H_i,\xi] = \alpha_i(H_i)\xi = |\alpha_i|\xi$. A straightforward calculation shows that the covariant derivatives of these left-invariant vector fields are given by $\nabla_\xi \xi = |\alpha_i| H_i$, $\nabla_\xi H_i = -|\alpha_i|\xi$, $\nabla_{H_i} H_i = 0$ and $\nabla_{H_i}\xi = 0$. Thus ξ and H_i, now considered as left-invariant vector fields, span an autoparallel subbundle of the tangent bundle of AN, and it follows that the orbit of the corresponding connected subgroup of AN through o is a totally geodesic real hyperbolic plane $\mathbb{R}H^2 \subset AN = M$. Let $\gamma : \mathbb{R} \to M$ be the geodesic in M with $\gamma(0) = o$ and $\dot\gamma(0) = \xi$. Clearly, this geodesic lies in the totally geodesic $\mathbb{R}H^2$. Using again the Koszul formula for the Levi Civita connection of AN it is easy to see that the tangent vector field $\dot\gamma$ of γ satisfies $\dot\gamma(t) = \frac{1}{\cosh(|\alpha_i|t)}\xi_{\gamma(t)} - \tanh(|\alpha_i|t)(H_i)_{\gamma(t)}$ for all $t \in \mathbb{R}$, where we view ξ and H_i as left-invariant vector fields on AN.

Let $t \in \mathbb{R}$, $g = \gamma(t) \in AN$, and denote by $I_{g^{-1}}$ the conjugation on G^o by g^{-1}. Clearly, $I_{g^{-1}}$ leaves AN invariant, and hence $I_{g^{-1}}(S_\xi)$ is also a subgroup of AN. Since $I_{g^{-1}}(S_\xi)\cdot o = g^{-1}S_\xi g \cdot o = \gamma(t)^{-1}S_\xi \cdot \gamma(t)$, we see that the orbit of the action of $I_{g^{-1}}(S_\xi)$ through o is the left translate from $\gamma(t)$ to o of the orbit of the action of S_ξ through $\gamma(t)$. Since $\dot\gamma(t)$ is a unit normal vector of $S_\xi \cdot \gamma(t)$ at $\gamma(t)$, and left translation $L_{g^{-1}}$ with g^{-1} is an isometry, the vector $\xi_t = L_{g^{-1}*}\dot\gamma(t) = \frac{1}{\cosh(|\alpha_i|t)}\xi_o - \tanh(|\alpha_i|t)(H_i)_o$ is a unit normal vector of $I_{g^{-1}}(S_\xi)\cdot o$ at o. It follows that $\mathrm{Ad}(g^{-1})\mathfrak{s}_\xi = \mathfrak{s}_{\xi_t}$, or equivalently, $\mathrm{Ad}(g)\mathfrak{s}_{\xi_t} = \mathfrak{s}_\xi$, where $\mathfrak{s}_{\xi_t}$ is the subalgebra of $\mathfrak{a}+\mathfrak{n}$ given by $\mathfrak{s}_{\xi_t} = (\mathfrak{a} + \mathfrak{n}) \ominus \mathbb{R}\xi_t$. Thus we have proved:

Lemma 4.2. *Let $\mathfrak{F}_i = \mathfrak{F}_\xi$ with some unit vector $\xi \in \mathfrak{g}_{\alpha_i}$. Then the leaf of $\mathfrak{F}_i$ at oriented distance $t \in \mathbb{R}$ in direction of ξ is isometrically congruent the orbit $S_{\xi_t} \cdot o$, where*

$$\xi_t = \frac{1}{\cosh(|\alpha_i|t)}\xi - \frac{1}{|\alpha_i|}\tanh(|\alpha_i|t)H_{\alpha_i}$$

and S_{ξ_t} is the connected Lie subgroup of AN with Lie algebra $(\mathfrak{a} + \mathfrak{n}) \ominus \mathbb{R}\xi_t$.

This lemma is very useful for studying the geometry of the leaves of $\mathfrak{F}_i$. Let A_{ξ_t} be the shape operator with respect to ξ_t of the orbit $S_{\xi_t} \cdot o$ at o. By means of the Weingarten formula for the shape operator and the Koszul formula for the Levi Civita connection of AN we have $2\langle A_{\xi_t}X, Y\rangle = 2\langle \nabla_X Y, \xi_t\rangle = \langle [X,Y], \xi_t\rangle - \langle [Y, \xi_t], X\rangle + \langle [\xi_t, X], Y\rangle$. Since $\mathfrak{s}_{\xi_t}$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$, we have $[X,Y] \in \mathfrak{s}_{\xi_t}$ and hence $\langle [X,Y], \xi_t\rangle = 0$ for all $X, Y \in \mathfrak{s}_{\xi_t}$. Moreover, since the adjoint transformation $\mathrm{ad}(\xi_t)^*$ of $\mathrm{ad}(\xi_t)$ satisfies $\mathrm{ad}(\xi_t)^* = -\mathrm{ad}(\theta\xi_t)$, we have $\langle [Y, \xi_t], X\rangle = -\langle \mathrm{ad}(\xi_t)Y, X\rangle = -\langle \mathrm{ad}(\xi_t)^* X, Y\rangle = \langle \mathrm{ad}(\theta\xi_t)X, Y\rangle$. Altogether we now easily get $\langle A_{\xi_t}X, Y\rangle = \frac{1}{2}(\langle \mathrm{ad}(\xi_t)X, Y\rangle + \langle X, \mathrm{ad}(\xi_t)Y\rangle) = \frac{1}{2}\langle (\mathrm{ad}(\xi_t) - \mathrm{ad}(\theta\xi_t))X, Y\rangle$ for all $X, Y \in \mathfrak{s}_{\xi_t} = T_o(S_{\xi_t} \cdot o)$. Thus we have proved:

Lemma 4.3. *Let $\mathfrak{F}_i = \mathfrak{F}_\xi$ with some unit vector $\xi \in \mathfrak{g}_{\alpha_i}$. The shape operator A_{ξ_t} with respect to ξ_t of the orbit $S_{\xi_t} \cdot o$ at o is given by*

$$A_{\xi_t}X = \left[\frac{1}{2\cosh(|\alpha_i|t)}(\xi - \theta\xi) - \frac{1}{|\alpha_i|}\tanh(|\alpha_i|t)H_{\alpha_i}\ ,\ X\right]_{\mathfrak{s}_{\xi_t}}$$

for all $X \in \mathfrak{s}_{\xi_t}$, where the subscript $\mathfrak{s}_{\xi_t}$ denotes the orthogonal projection onto $\mathfrak{s}_{\xi_t}$.

We will now investigate the principal curvatures of the leaves of the foliation $\mathfrak{F}_i$. Let $\mathfrak{F}_i = \mathfrak{F}_\xi$ with some unit vector $\xi \in \mathfrak{g}_{\alpha_i}$. We identify the leaf of $\mathfrak{F}_i$ at distance t in direction ξ with $S_{\xi_t} \cdot o$.

Proposition 4.4. *Let $\mathfrak{F}_i = \mathfrak{F}_\xi$ with some unit vector $\xi \in \mathfrak{g}_{\alpha_i}$. For the principal curvatures of $S_{\xi_t} \cdot o$ we have:*

(1) *The $(r-1)$-dimensional subspace $\mathfrak{a} \ominus \mathbb{R}H_{\alpha_i}$ is invariant under A_{ξ_t} and the corresponding principal curvature is* 0.

(2) *The subspace* $(\mathfrak{g}_{\alpha_i} \ominus \mathbb{R}\xi) \oplus \mathfrak{g}_{2\alpha_i}$ *is invariant under* A_{ξ_t} *and the corresponding principal curvatures are* $-|\alpha_i|\tanh(|\alpha_i|t)$ *with multiplicity* $\dim \mathfrak{g}_{\alpha_i} - \dim \mathfrak{g}_{2\alpha_i} - 1$ *and*

$$-\frac{3}{2}|\alpha_i|\tanh(|\alpha_i|t) \pm \frac{1}{2}|\alpha_i|\sqrt{2-\tanh^2(|\alpha_i|t)}$$

with multiplicity $\dim \mathfrak{g}_{2\alpha_i}$.

(3) *The one-dimensional subspace* $(\mathbb{R}H_{\alpha_i} \oplus \mathbb{R}\xi) \ominus \mathbb{R}\xi_t$ *is invariant under* A_{ξ_t} *with corresponding principal curvature* $-|\alpha_i|\tanh(|\alpha_i|t)$.

(4) *Define* k *in the normalizer* $N_K(\mathfrak{a})$ *of* $\mathfrak{a}$ *in* K *by*

$$k = \operatorname{Exp}\left(\frac{\pi}{\sqrt{2}|\alpha_i|}(\xi + \theta\xi)\right).$$

The subspace $\mathfrak{n} \ominus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i})$ *is invariant under* A_{ξ_t} *and* $\operatorname{Ad}(k)$ *and we have* $A_{\xi_t}\operatorname{Ad}(k)X = -\operatorname{Ad}(k)A_{\xi_t}X$ *for all* $X \in \mathfrak{n} \ominus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i})$. *In particular, if* $A_{\xi_t}X = cX$, *then* $A_{\xi_t}\operatorname{Ad}(k)X = -c\operatorname{Ad}(k)X$.

Proof.

(1): For all $H \in \mathfrak{a} \ominus \mathbb{R}H_{\alpha_i}$ we have $[\xi, H] = -\alpha_i(H)\xi = 0$, $[\theta\xi, H] \in \mathfrak{g}_{-\alpha_i}$ and $[H_{\alpha_i}, H] = 0$, which implies $A_{\xi_t}H = 0$ by means of Lemma 4.3.

(2): We have $\langle[\theta\xi, X], H\rangle = \langle X, [H,\xi]\rangle = \alpha_i(H)\langle X, \xi\rangle = 0$ for all $X \in \mathfrak{g}_{\alpha_i} \ominus \mathbb{R}\xi$ and $H \in \mathfrak{a}$. Since obviously $[\theta\xi, X] \in \mathfrak{g}_0$, this implies $[\theta\xi, \mathfrak{g}_{\alpha_i} \ominus \mathbb{R}\xi]_{\mathfrak{s}_{\xi_t}} = 0$, and shows that $(\mathfrak{g}_{\alpha_i} \ominus \mathbb{R}\xi) \oplus \mathfrak{g}_{2\alpha_i}$ is invariant under A_{ξ_t}. The vector space $\mathfrak{g}_{\alpha_i} \ominus \mathbb{R}\xi$ can be decomposed orthogonally into $\mathfrak{g}_{\alpha_i} \ominus \mathbb{R}\xi = V_1 \oplus V_2$, where V_1 is the kernel of the linear map $\operatorname{ad}(\xi)|\mathfrak{g}_{\alpha_i} : \mathfrak{g}_{\alpha_i} \to \mathfrak{g}_{2\alpha_i}$ and V_2 is the image of the linear map $\operatorname{ad}(\theta\xi)|\mathfrak{g}_{2\alpha_i} : \mathfrak{g}_{2\alpha_i} \to \mathfrak{g}_{\alpha_i}$. One can easily see that V_1 consists of principal curvature vectors with principal curvature $-|\alpha_i|\tanh(|\alpha_i|t)$ and corresponding multiplicity $\dim \mathfrak{g}_{\alpha_i} - \dim \mathfrak{g}_{2\alpha_i} - 1$. Note that $\operatorname{ad}(\xi)|V_2 : V_2 \to \mathfrak{g}_{2\alpha_i}$ is an isomorphism because of $[\xi, [\theta\xi, Y]] = -[Y, [\xi, \theta\xi]] = [Y, H_{\alpha_i}] = -2|\alpha_i|^2 Y$ for all $Y \in \mathfrak{g}_{2\alpha_i}$. Moreover, if $Y \in \mathfrak{g}_{2\alpha_i}$ is a unit vector, then $X := \frac{1}{\sqrt{2}|\alpha_i|}[\theta\xi, Y] \in \mathfrak{g}_{\alpha_i}$ is also a unit vector. It follows easily from Lemma 4.3 that the two-dimensional linear subspace $\mathbb{R}X \oplus \mathbb{R}Y$ of $(\mathfrak{g}_{\alpha_i} \ominus \mathbb{R}\xi) \oplus \mathfrak{g}_{2\alpha_i}$ is invariant under A_{ξ_t} and has the matrix representation

$$\begin{pmatrix} -|\alpha_i|\tanh(|\alpha_i|t) & -\frac{|\alpha_i|}{\sqrt{2}\cosh(|\alpha_i|t)} \\ -\frac{|\alpha_i|}{\sqrt{2}\cosh(|\alpha_i|t)} & -2|\alpha_i|\tanh(|\alpha_i|t) \end{pmatrix}$$

with respect to X and Y. From this one can calculate the eigenvalues directly.

(3): The subspace $(\mathbb{R}H_{\alpha_i} \oplus \mathbb{R}\xi) \ominus \mathbb{R}\xi_t$ is of dimension one. Since ξ and $\frac{1}{|\alpha_i|}H_{\alpha_i}$ are orthonormal, the vector $X := \tanh(|\alpha_i|t)\xi + \frac{1}{|\alpha_i|\cosh(|\alpha_i|t)}H_{\alpha_i}$ belongs to this subspace. A direct calculation using Lemma 4.3 shows that $A_{\xi_t}X = -|\alpha_i|\tanh(|\alpha_i|t)X$.

(4): The subspace $\mathbb{R}H_{\alpha_i} \oplus \mathbb{R}\xi \oplus \mathbb{R}\theta\xi$ is a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$. The isometry $k = \mathrm{Exp}\left(\frac{\pi}{\sqrt{2}|\alpha_i|}(\xi + \theta\xi)\right)$ is in $N_K(\mathfrak{a})$ and satisfies $\mathrm{Ad}(k)(H_{\alpha_i}) = -H_{\alpha_i}$ and $\mathrm{Ad}(k)(\xi - \theta\xi) = \theta\xi - \xi$. The significance of k is that $\mathrm{Ad}(k)|\mathfrak{a}$ is the reflection in the hyperplane $\mathfrak{a} \ominus \mathbb{R}H_{\alpha_i}$. It is clear that $\mathrm{Ad}(k)$ preserves $\mathfrak{n} \ominus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i})$, and from (1), (2) and (3) it follows that A_{ξ_t} preserves this subspace as well. Moreover, from the explicit expression of A_{ξ_t} in Lemma 4.3 we easily see that $\mathrm{Ad}(k) \circ A_{\xi_t} = -\mathrm{Ad}(k) \circ A_{\xi_t}$ holds on $\mathfrak{n} \ominus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i})$. q.e.d.

This proposition shows that the sum of the principal curvatures on $\mathfrak{n} \ominus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i})$ is zero. This makes it easy to calculate the mean curvature of each leaf of the foliation $\mathfrak{F}_i$. The following corollary is a direct consequence of Proposition 4.4:

Corollary 4.5. *The (constant) mean curvature μ_t of the leaf of $\mathfrak{F}_i$ at distance t in direction ξ satisfies*

$$\mu_t = -\frac{|\alpha_i|\tanh(|\alpha_i|t)}{n-1}(\dim \mathfrak{g}_{\alpha_i} + 2\dim \mathfrak{g}_{2\alpha_i}).$$

Therefore the leaf of $\mathfrak{F}_i$ through o, i.e., the orbit $S_\xi \cdot o$, is the only minimal leaf of $\mathfrak{F}_i$. This minimal leaf is an austere submanifold, i.e., if c is a principal curvature then also $-c$ is a principal curvature with the same multiplicity.

Each foliation $\mathfrak{F}_i$ has a kind of reflective symmetry. Let $\mathfrak{F}_i = \mathfrak{F}_\xi$ with a unit vector $\xi \in \mathfrak{g}_{\alpha_i}$. Consider the gradation $\mathfrak{g} = \bigoplus_{k=-m}^{m} \mathfrak{g}^k$ of $\mathfrak{g}$ and the subalgebra $\mathfrak{h} = \bigoplus_{k \text{ even}} \mathfrak{g}^k$ of $\mathfrak{g}$. Let H be the connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. Since $\mathfrak{h}$ is invariant under the Cartan involution θ, the orbit $Q = H \cdot o$ is a totally geodesic submanifold of M. The subspace $\nu_o Q = \bigoplus_{k \text{ odd}} \mathfrak{p}_k \subset \mathfrak{p}$ with $\mathfrak{p}_k = (\mathfrak{g}^k \oplus \mathfrak{g}^{-k}) \cap \mathfrak{p}$ is the normal space of Q at o. It is easy to see that $\nu_o Q$ is a Lie triple system in $\mathfrak{p}$, which implies that for each normal space of Q there exists a totally geodesic submanifold of M that is tangent to that normal space. This implies that the reflection Φ_Q of M in Q is an isometry. By construction, $\mathrm{Ad}(\Phi_Q)X = X$ if $X \in \bigoplus_{k \text{ even}} \mathfrak{g}^k$, and $\mathrm{Ad}(\Phi_Q)X = -X$ if $X \in \bigoplus_{k \text{ odd}} \mathfrak{g}^k$. Thus $\mathfrak{s}_\xi$ is invariant under $\mathrm{Ad}(\Phi_Q)$ and it follows that Φ_Q maps $S_\xi \cdot o$ onto itself. Since $\mathrm{Ad}(\Phi_Q)\xi = -\xi$, we then conclude that Φ_Q interchanges the two orbits at any positive distance to the minimal orbit. It is worthwhile to point out that the submanifold Q is independent of the choice of the simple root α_i and the choice of the unit vector $\xi \in \mathfrak{g}_{\alpha_i}$. We thus have proved:

Proposition 4.6. *There exists a totally geodesic submanifold Q of M such that the reflection Φ_Q of M in Q is an isometry that leaves the minimal leaf of $\mathfrak{F}_i$ invariant and interchanges the two leaves in $\mathfrak{F}_i$ at any given positive distance to the minimal leaf.*

Next we shall compare the principal curvatures of the leaves of $\mathfrak{F}_i$ and $\mathfrak{F}_j$. As Corollary 4.5 shows, the mean curvature of the leaves of $\mathfrak{F}_i$ depends only on the length of the root α_i and the distance t to the unique minimal leaf of $\mathfrak{F}_i$. This is also true for the principal curvatures, as the following result shows:

Theorem 4.7. *If $\alpha_i, \alpha_j \in \Lambda$ with $|\alpha_i| = |\alpha_j|$, then the foliations $\mathfrak{F}_i$ and $\mathfrak{F}_j$ have the same principal curvatures, counted with multiplicities. More precisely, let $\xi \in \mathfrak{g}_{\alpha_i}$ and $\eta \in \mathfrak{g}_{\alpha_j}$ be unit vectors. The geodesic $\gamma_\xi : \mathbb{R} \to M$ with $\gamma_\xi(0) = o$ and $\dot\gamma_\xi(0) = \xi$ intersects each leaf of $\mathfrak{F}_\xi$ and $\dot\gamma_\xi(t)$ is a unit normal vector of the leaf $\mathfrak{F}_\xi(t)$ of $\mathfrak{F}_\xi$ through $\gamma_\xi(t)$, which by homogeneity of the leaves naturally extends to a unit normal vector field $\xi^{(t)}$ on the leaf $\mathfrak{F}_\xi(t)$ (and analogously for η). Then the principal curvatures, counted with multiplicities, of $\mathfrak{F}_\xi(t)$ with respect to $\xi^{(t)}$ and of $\mathfrak{F}_\eta(t)$ with respect to $\eta^{(t)}$ coincide.*

Proof. Let $\xi \in \mathfrak{g}_{\alpha_i}$ and $\eta \in \mathfrak{g}_{\alpha_j}$ be unit vectors. We have to show that for each $t \in \mathbb{R}$ the orbits $S_{\xi_t} \cdot o$ and $S_{\eta_t} \cdot o$ have the same principal curvatures, counted with multiplicities. Since α_i and α_j have the same length there exists an isometry $k \in N_K(\mathfrak{a})$ for which the induced action $\mathrm{Ad}(k)^*$ of $\mathrm{Ad}(k)$ on Σ maps α_i to α_j, and hence $\mathrm{Ad}(k)\xi \in \mathfrak{g}_{\alpha_j}$. If $\dim \mathfrak{g}_{\alpha_j} > 1$, we can assume $\mathrm{Ad}(k)\xi = \eta$ by Exercise 2 on p. 211 in [11]. If $\dim \mathfrak{g}_{\alpha_j} = 1$, we can assume $\mathrm{Ad}(k)\xi = \eta$ by Proposition 4.6. Note that, by construction, $\mathrm{Ad}(k)H_{\alpha_i} = H_{\alpha_j}$ and $\mathrm{Ad}(k)\xi_t = \eta_t$. The map $\mathrm{Ad}(k)$ induces a linear isometry from $(\mathfrak{a} \oplus \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i}) \ominus \mathbb{R}\xi_t$ onto $(\mathfrak{a} \oplus \mathfrak{g}_{\alpha_j} \oplus \mathfrak{g}_{2\alpha_j}) \ominus \mathbb{R}\eta_t$, and using Lemma 4.3 it is easy to see that $\mathrm{Ad}(k) \circ A_{\xi_t} = A_{\eta_t} \circ \mathrm{Ad}(k)$ holds on this subspace. Let $\lambda \in \Sigma^+$ so that $\lambda - \alpha_i \notin \Sigma^+$, and let V_{λ,α_i} be the linear subspace of $\mathfrak{n}$ spanned by the root spaces of the roots in the α_i-string of λ. It follows from Lemma 4.3

that V_{λ,α_i} is invariant under A_{ξ_t}. If $\mathrm{Ad}(k)^*\lambda \in \Sigma^+$, then, using again Lemma 4.3, it follows that $\mathrm{Ad}(k) \circ A_{\xi_t} = A_{\eta_t} \circ \mathrm{Ad}(k)$ holds on V_{λ,α_i}. If $\mathrm{Ad}(k)^*\lambda \in \Sigma^- = \Sigma \setminus \Sigma^+$, we define $k' = \mathrm{Exp}\left(\frac{\pi}{\sqrt{2}|\alpha_j|}(\eta + \theta\eta)\right)$ (see Proposition 4.4 (4)) and a linear isometry F from V_{λ,α_i} into $\mathfrak{n}$ by $F = \mathrm{Ad}(k') \circ \theta \circ \mathrm{Ad}(k)|V_{\lambda,\alpha_i}$. Then, by construction, $F(H_{\alpha_i}) = H_{\alpha_j}$, $F(\xi) = \eta$, $F(\theta\xi) = \theta\eta$, and hence also $F(\xi_t) = \eta_t$. A straightforward calculation shows that $F \circ A_{\xi_t} = A_{\eta_t} \circ F$ holds on V_{λ,α_i}. Since $\mathfrak{n} \ominus (\mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{2\alpha_i})$ is spanned by subspaces of the form V_{λ,α_i} this finishes the proof. q.e.d.

We will now investigate the congruence problem for the foliations $\mathfrak{F}_i$. First, let $\alpha, \beta \in \Lambda$ be two distinct simple roots and $\xi \in \mathfrak{g}_\alpha$, $\eta \in \mathfrak{g}_\beta$ be two unit vectors. Assume that $\mathfrak{F}_\xi$ and $\mathfrak{F}_\eta$ are isometrically congruent. It is clear that an isometry mapping $\mathfrak{F}_\xi$ to $\mathfrak{F}_\eta$ maps the unique minimal leaf $S_\xi \cdot o$ of $\mathfrak{F}_\xi$ to the unique minimal leaf $S_\eta \cdot o$ of $\mathfrak{F}_\eta$. Thus there exists an isometry $k \in K$ with $k(S_\xi \cdot o) = S_\eta \cdot o$. It is also clear that k preserves the mean curvature of each leaf as well as the distance of any leaf to the minimal leaf. Therefore, using the explicit expression in Corollary 4.5 of the mean curvature of the leaves, we necessarily have $|\alpha| = |\beta|$. We conclude that if α and β are simple roots with different length, then the foliations $\mathfrak{F}_\xi$ and $\mathfrak{F}_\eta$ cannot be isometrically congruent.

We now assume that α and β have the same length and that $S_\xi \cdot o$ and $S_\eta \cdot o$ are congruent under an isometry of M. Our next aim is to prove that there exists a symmetry of the Dynkin diagram of the restricted root system of M mapping α to β. First of all, a similar proof as in Lemma 3.3 shows that S_ξ is completely solvable, that is, $\mathrm{Ad}(S_\xi)$ can be represented by upper triangular matrices in a suitable basis of the Lie algebra $\mathfrak{s}_\xi$. By assumption, $S_\xi \cdot o$ and $S_\eta \cdot o$ are isometric to each other. Since any two completely solvable transitive groups of isometries on a Riemannian manifold are conjugate to each other in the full isometry group of that manifold (see, e.g., [1]), we conclude that the Lie algebras $\mathfrak{s}_\xi$ and $\mathfrak{s}_\eta$ are isomorphic.

Let $F : \mathfrak{s}_\xi \to \mathfrak{s}_\eta$ be a Lie algebra isomorphism. It is easy to see that $\mathfrak{a}$ is a Cartan subalgebra of both $\mathfrak{s}_\xi$ and $\mathfrak{s}_\eta$. Since any two Cartan subalgebras of a solvable Lie algebra are conjugate to each other under an inner automorphism we can assume without loss of generality that $F(\mathfrak{a}) = \mathfrak{a}$. If $\dim \mathfrak{g}_\alpha > 1$, then Σ^+ is the set of roots of $\mathfrak{s}_\xi$ with respect to $\mathfrak{a}$, and if $\dim \mathfrak{g}_\alpha = 1$, then $\Sigma^+ \setminus \{\alpha\}$ is the set of roots of $\mathfrak{s}_\xi$ with respect to $\mathfrak{a}$. It follows that if $\dim \mathfrak{g}_\alpha > 1$, then also $\dim \mathfrak{g}_\beta > 1$, and if $\dim \mathfrak{g}_\alpha = 1$, then also $\dim \mathfrak{g}_\beta = 1$. We will now distinguish the two cases when $\dim \mathfrak{g}_\alpha > 1$ and $\dim \mathfrak{g}_\alpha = 1$.

Case 1. $\dim \mathfrak{g}_\alpha > 1$. Then we get $[F(H), F(X_\lambda)] = F([H, X_\lambda]) = \lambda(H)F(X_\lambda)$ for all $H \in \mathfrak{a}$ and $X_\lambda \in \mathfrak{g}_\lambda \cap \mathfrak{s}_\xi$, $\lambda \in \Sigma^+$. Note that $\mathfrak{g}_\lambda \cap \mathfrak{s}_\xi = \mathfrak{g}_\lambda$ for all $\lambda \in \Sigma^+ \setminus \{\alpha\}$ and $\mathfrak{g}_\alpha \cap \mathfrak{s}_\xi = \mathfrak{g}_\alpha \ominus \mathbb{R}\xi$. Thus, for all $\lambda \in \Sigma^+ \setminus \{\alpha\}$, F maps the root space $\mathfrak{g}_\lambda$ onto the root space $\mathfrak{g}_{F^*(\lambda)} \cap \mathfrak{s}_\eta$ with $F^*(\lambda) = \lambda \circ F^{-1}|\mathfrak{a}$. Moreover, F maps $\mathfrak{g}_\alpha \ominus \mathbb{R}\xi$ onto $\mathfrak{g}_{F^*(\alpha)} \cap \mathfrak{s}_\eta$, and consequently F^* induces a permutation of the roots in Σ^+. Let $\lambda, \mu \in \Lambda$ be simple roots and let $\lambda, \ldots, \lambda + q\mu$ be the μ-string containing λ. Since $F^* : \Sigma^+ \to \Sigma^+$ is a bijection, and as $F^*(\lambda+\mu) = F^*(\lambda)+F^*(\mu)$, the roots $F^*(\lambda), \ldots, F^*(\lambda) + qF^*(\mu)$ form the $F^*(\mu)$-string containing $F^*(\lambda)$. It follows that F^* preserves the edge and arrow relations between the vertices in the Dynkin diagram of Σ, that is, F^* is a symmetry of the Dynkin diagram of Σ. In particular, F^* maps simple roots to simple roots, that is, $F^*(\Lambda) = \Lambda$. Taking into account the multiplicities of simple roots it now follows easily that $F(\mathfrak{g}_\alpha \ominus \mathbb{R}\xi) = \mathfrak{g}_\beta \ominus \mathbb{R}\eta$, and hence $F^*(\alpha) = \beta$. We thus have proved that if $\dim \mathfrak{g}_\alpha > 1$, and if $S_\xi \cdot o$ is isometric to $S_\eta \cdot o$, then there exists a symmetry of the Dynkin diagram of the restricted root system Σ mapping the simple root α to the simple root β.

Case 2. $\dim \mathfrak{g}_\alpha = 1$. Recall that in this case the set of roots of $\mathfrak{s}_\xi$ and $\mathfrak{s}_\eta$ with respect to the Cartan subalgebra $\mathfrak{a}$ is $\Sigma^+ \setminus \{\alpha\}$ and $\Sigma^+ \setminus \{\beta\}$, respectively. As in the previous case we can construct a bijection F^* of $\Sigma^+ \setminus \{\alpha\}$ onto $\Sigma^+ \setminus \{\beta\}$ preserving the string relations between roots. Our aim now is to show that there exists a symmetry of the Dynkin diagram of Σ mapping α to β.

We can assume that $2\alpha \notin \Sigma^+$, because α is a simple root and if $2\alpha \in \Sigma^+$ we have $\Sigma = BC_r$, but if $\Lambda \subset BC_r$ there exists only one simple root $\alpha \in \Lambda$ with $2\alpha \in \Sigma^+$ and the length of it is different from the length of all other simple roots in Λ.

We now define $\Lambda_\alpha = \{\gamma + q\alpha \mid \gamma \in \Lambda \setminus \{\alpha\}\ ,\ \gamma + q\alpha \in \Sigma^+\ ,\ q \in \{0, 1, 2, 3\}\}$. First of all, from the structure of $\mathfrak{n}$ as a graded Lie algebra $\mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_m$ with $\mathfrak{n}_1 = \mathfrak{g}_{\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{\alpha_r}$ it is clear that $\mathfrak{n} \ominus \mathfrak{g}_\alpha$ is a graded Lie algebra generated by $\mathrm{Gen}(\mathfrak{n} \ominus \mathfrak{g}_\alpha) = \bigoplus_{\gamma \in \Lambda_\alpha} \mathfrak{g}_\gamma$. Moreover, if we denote by m_α the coefficient of α in the maximal root $\widetilde{\alpha}$ in Σ^+, we see that $\mathfrak{n} \ominus \mathfrak{g}_\alpha$ is an $(m - m_\alpha)$-step nilpotent Lie algebra.

We will now associate to each such subalgebra $\mathfrak{n} \ominus \mathfrak{g}_\alpha$ a diagram in the following manner. Each vertex $v \in \Lambda_\alpha$ with $2v \notin \Sigma^+$ is represented by $\circ$, and each vertex $v \in \Lambda_\alpha$ with $2v \in \Sigma^+$ is represented by $\circledcirc$. Then connect two vertices with each other if one of the following cases occurs:

$$\begin{array}{ll}
\underset{v_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{v_2}{\circ} & \text{if } v_1 + v_2 \in \Sigma^+,\ 2v_1 + v_2, v_1 + 2v_2 \notin \Sigma^+; \\
\underset{v_1}{\circ}\!\!\Longrightarrow\!\!\underset{v_2}{\circ} & \text{if } v_1 + v_2, v_1 + 2v_2 \in \Sigma^+,\ 2v_1 + v_2, v_1 + 3v_2 \notin \Sigma^+; \\
\underset{v_1}{\circ}\!\!\Rrightarrow\!\!\underset{v_2}{\circ} & \text{if } v_1 + v_2, v_1 + 2v_2, v_1 + 3v_2 \in \Sigma^+,\ 2v_1 + v_2 \notin \Sigma^+; \\
\underset{v_1}{\circ}\!\!\Longleftrightarrow\!\!\underset{v_2}{\circledcirc} & \text{if } v_1 + v_2 \in \Sigma^+; \\
\underset{v_1}{\circledcirc}\!\!-\!\!\!-\!\!\!-\!\!\underset{v_2}{\circledcirc} & \text{if } v_1 + v_2 \in \Sigma^+.
\end{array}$$

In the case $\circ\!\!\Longleftrightarrow\!\!\circledcirc$ we have $2v_1+v_2, v_1+2v_2 \in \Sigma^+$ and $3v_1+v_2, v_1+3v_2 \notin \Sigma^+$, which explains the two lines and the two arrows. In the case $\circledcirc\!\!-\!\!\!-\!\!\!-\!\!\circledcirc$ we have $v_1 + 2v_2, 2v_1 + v_2 \notin \Sigma^+$, which explains that there is only one line. It follows from general properties of root systems that $v_1+v_2 \notin \Sigma^+$ if none of the above cases occurs. For $i \leq j$ we define

$$\underset{v_i}{\circ}\cdots\cdots\underset{v_j}{\circ} = \begin{cases} \underset{v_i}{\circ} & \text{if } i = j \\ \underset{v_i}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{v_{i+1}}{\circ}\ ,\ \ldots\ ,\ \underset{v_{j-1}}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{v_j}{\circ} & \text{if } i < j. \end{cases}$$

Since F maps the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{s}_\xi$ onto the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{s}_\eta$, it induces a transformation from Λ_α onto Λ_β such that the string relations among the elements in Λ_α are the same as the string relations among the corresponding elements in Λ_β. It follows that the diagrams of $\mathfrak{n}\ominus\mathfrak{g}_\alpha$ and $\mathfrak{n}\ominus\mathfrak{g}_\beta$ are isomorphic. Thus the diagrams provide us with a simple tool for deciding nonisomorphy of two Lie algebras $\mathfrak{s}_\xi$ and $\mathfrak{s}_\eta$.

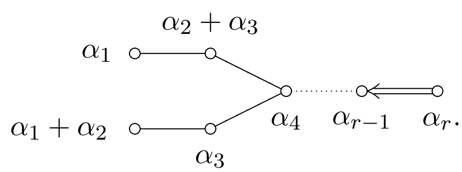
By taking into account the following two observations we can reduce the number of diagrams that have to be drawn. Firstly, we must have $\dim \mathrm{Gen}(\mathfrak{n} \ominus \mathfrak{g}_\alpha) = \dim \mathrm{Gen}(\mathfrak{n} \ominus \mathfrak{g}_\beta)$. It is quite easy to determine $\dim \mathrm{Gen}(\mathfrak{n}\ominus\mathfrak{g}_\alpha)$ from the Dynkin diagram. First determine the number all vertices that are not connected with α. For each other vertex add the number two if it is connected with α by one line or by more lines with the arrow pointing away from α, and add the number three or four if it is connected with α by two or three lines with the arrow pointing towards α. Secondly, the number of steps of $\mathfrak{n} \ominus \mathfrak{g}_\alpha$ and $\mathfrak{n} \ominus \mathfrak{g}_\beta$ must be equal, that is, $m_\alpha = m_\beta$. Taking into account these two simple criteria and the symmetries of the Dynkin diagram we are left with the following cases:

1. $\Sigma = A_r$, $r \geq 5$ and $\alpha, \beta \in \{\alpha_2, \ldots, \alpha_s\}$ with $s = \left[\frac{r+1}{2}\right]$;

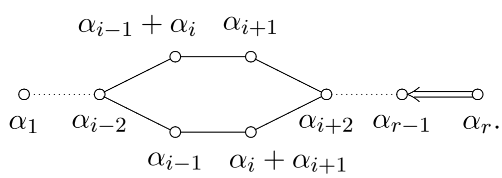
2. $\Sigma = B_r$, $r \geq 4$ and $\alpha, \beta \in \{\alpha_2, \dots, \alpha_{r-1}\}$;
3. $\Sigma = C_r$, $r \geq 5$ and $\alpha, \beta \in \{\alpha_2, \dots, \alpha_{r-2}\}$;
4. $\Sigma = D_r$, $r \geq 5$ and $\alpha, \beta \in \{\alpha_1, \alpha_r\}$, and $r \geq 6$ and $\alpha, \beta \in \{\alpha_2, \dots, \alpha_{r-3}\}$;
5. $\Sigma = E_7$ and $\alpha, \beta \in \{\alpha_1, \alpha_2\}$ or $\alpha, \beta \in \{\alpha_3, \alpha_5\}$;
6. $\Sigma = E_8$ and $\alpha, \beta \in \{\alpha_1, \alpha_8\}$ or $\alpha, \beta \in \{\alpha_3, \alpha_6\}$;
7. $\Sigma = BC_r$, $r \geq 4$ and $\alpha, \beta \in \{\alpha_1, \dots, \alpha_{r-2}\}$.

It is a straightforward but lengthy exercise to draw all the relevant diagrams. We omit the list of all diagrams here and just illustrate it by discussing Case 3:

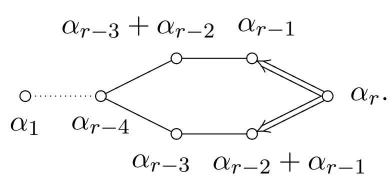
a) $\mathfrak{n} \ominus \mathfrak{g}_{\alpha_2}$:



b) $\mathfrak{n} \ominus \mathfrak{g}_{\alpha_i}$, $3 \leq i \leq r-3$, $r \geq 6$:



c) $\mathfrak{n} \ominus \mathfrak{g}_{\alpha_{r-2}}$:



By comparing these diagrams it is now easy to conclude that there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\alpha) = \beta$. The other cases can be settled in a similar fashion.

Conversely, let $\alpha, \beta \in \Lambda$ and $\xi \in \mathfrak{g}_\alpha$ and $\eta \in \mathfrak{g}_\beta$ be unit vectors. Assume that there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\alpha) = \beta$. Then there exists an isometry k in the normalizer of $\mathfrak{a} + \mathfrak{n}$ in K such that the induced action of $F = \mathrm{Ad}(k) \in \mathrm{Aut}(\mathfrak{g})$ on Σ^+ is equal to the action of P on Σ^+. Then $F(\mathfrak{s}_\xi) = \mathfrak{s}_{\eta'}$ for some unit vector $\eta' \in \mathfrak{g}_\beta$. Using Lemma 4.1 we see that the foliations $\mathfrak{F}_\xi$ and $\mathfrak{F}_\eta$ are isometrically congruent. Thus we have proved:

Theorem 4.8. *Two foliations $\mathfrak{F}_i$ and $\mathfrak{F}_j$ are isometrically congruent to each other if and only if there exists a symmetry $P \in \mathrm{Aut}(DD)$ with $P(\alpha_i) = \alpha_j$.*

This result has a remarkable consequence. A smooth function $f : M \to \mathbb{R}$ is isoparametric if there exist smooth real-valued functions a, b such that $||\mathrm{grad}\, f||^2 = (a \circ f)f$ and $\Delta f = (b \circ f)f$. The interest in such functions stems from geometrical optics. The first condition means that the level sets of f are equidistant, and then the second condition is equivalent to the constancy of the mean curvature of the level sets. The level sets of an isoparametric function form an isoparametric system on M. Clearly, the level sets of a cohomogeneity one action form an isoparametric system, where the corresponding isoparametric function is just the natural projection onto the orbit space. Such isoparametric systems are called homogeneous. From Theorems 4.7 and 4.8 we conclude:

Corollary 4.9. *On every connected irreducible Riemannian symmetric space of noncompact type and rank ≥ 3 there exist homogeneous isoparametric systems with the same principal curvatures, counted with multiplicities.*

Ferus, Karcher and Münzner [9] discovered such a phenomenon for inhomogeneous isoparametric systems on spheres, but apparently it was not clear whether homogeneous isoparametric systems can be distinguished in general by their “spectral data”. We illustrate the corollary by an example.

Example. Let M be the set of all Euclidean structures on $\mathbb{R}^4$. Then M can be realized as the 9-dimensional noncompact symmetric space $\mathrm{SL}(4, \mathbb{R})/\mathrm{SO}(4)$ of rank three. We get an Iwasawa decomposition of $\mathrm{SL}(4, \mathbb{R})$ by choosing for A the diagonal (4×4)-matrices with determinant one, and for N the upper triangular (4×4)-matrices for which each element on the diagonal is equal to one. Then

$$M = AN = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \middle| x_{ij} \in \mathbb{R} \ , \ x_{11}x_{22}x_{33}x_{44} = 1 \right\}.$$

The Lie algebra $\mathfrak{a}$ of A is the 3-dimensional abelian Lie algebra of all diagonal (4×4)-matrices with real coefficients and trace zero, and the Lie algebra $\mathfrak{n}$ of N is the 6-dimensional 3-step nilpotent Lie algebra of all upper triangular (4×4)-matrices with real coefficients. We define three vectors $H_1, H_2, H_3 \in \mathfrak{a}$ by

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ H_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and denote by $\alpha_1, \alpha_2, \alpha_3$ the dual one-forms on $\mathfrak{a}$ with respect to the Killing form of $\mathfrak{sl}(4,\mathbb{R})$. Then $\Sigma^+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3\}$ is the set of positive restricted roots of $\mathfrak{sl}(4,\mathbb{R})$ with respect to $\mathfrak{a}$ and $\Lambda = \{\alpha_1, \alpha_2, \alpha_3\}$ is the corresponding set of simple roots. The resulting restricted root system is A_3 and its Dynkin diagram is $\underset{\alpha_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{\alpha_2}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{\alpha_3}{\circ}$. The simple root spaces $\mathfrak{g}_{\alpha_1}, \mathfrak{g}_{\alpha_2}, \mathfrak{g}_{\alpha_3}$ are spanned by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. We define three subgroups S_1, S_2, S_3 of AN by

$$S_1 = \left\{ \begin{pmatrix} x_{11} & 0 & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \;\middle|\; x_{ij} \in \mathbb{R} \ , \ x_{11}x_{22}x_{33}x_{44} = 1 \right\},$$

$$S_2 = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & 0 & x_{24} \\ 0 & 0 & x_{33} & x_{34} \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \;\middle|\; x_{ij} \in \mathbb{R} \ , \ x_{11}x_{22}x_{33}x_{44} = 1 \right\},$$

$$S_3 = \left\{ \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ 0 & x_{22} & x_{23} & x_{24} \\ 0 & 0 & x_{33} & 0 \\ 0 & 0 & 0 & x_{44} \end{pmatrix} \;\middle|\; x_{ij} \in \mathbb{R} \ , \ x_{11}x_{22}x_{33}x_{44} = 1 \right\}.$$

The corresponding subalgebras of $\mathfrak{a} + \mathfrak{n}$ are $\mathfrak{s}_i = (\mathfrak{a} + \mathfrak{n}) \ominus \mathfrak{g}_{\alpha_i}$. Since the Dynkin diagram has only one nontrivial symmetry, we see that the homogeneous isoparametric systems induced by the actions of S_1 and S_3 are congruent under an isometry of M. On the other hand, the homogeneous isoparametric systems induced by the actions of S_1 and S_2 are not congruent under an isometry of M, but have the same principal curvatures, counted with multiplicities.

5. The classification

By means of Theorems 3.5 and 4.8 we have solved the congruency problem on the parameter space $\mathbb{R}P^{r-1} \cup \{1, \ldots, r\}$. It remains to show that any homogeneous cohomogeneity one foliation on M is isometrically congruent to a foliation constructed from the parameter space.

Let M be a connected irreducible Riemannian symmetric space of noncompact type and $\mathfrak{F}$ a homogeneous foliation of codimension one on M. This means that there exists a connected subgroup S of G^o such that the leaves of $\mathfrak{F}$ coincide with the orbits of the action of S on M. Let $S = L \cdot R$ be a Levi decomposition of S, where L and R are the semisimple and solvable factor. Moreover, let $L = L_K \cdot L_{AN}$ be an Iwasawa decomposition of L, where L_K and L_{AN} are the compact and solvable factor. Then $S = L \cdot R = L_K \cdot (L_{AN} \cdot R)$. The compact group L_K has a fixed point o in M by Cartan's fixed point theorem. It follows that the solvable subgroup $L_{AN} \cdot R$ of S is transitive on $S \cdot o$, the leaf of $\mathfrak{F}$ through o, and consequently also on every leaf of $\mathfrak{F}$. Without loss of generality we can therefore assume that S is solvable.

Since S is solvable, we can decompose it into $S = T \cdot B$ with a compact subgroup T and a normal k-solvable subgroup B (see e.g., [19]). The latter means that there exists a compact normal subgroup B_K of B such that B/B_K is simply connected. The subgroup B_K is then maximal compact in B and contained in the center of B. An analogous argument as above shows that already B is transitive on each leaf of $\mathfrak{F}$. Since B is a subgroup of the isometry group of M, it acts effectively on M. Since the leaves of $\mathfrak{F}$ are of codimension one, B must also act effectively on each leaf of $\mathfrak{F}$. The isotropy subgroup of B at a point in M is compact and hence contained in some conjugate of B_K, and therefore lies in the center of B. Because of effectivity of B it follows that at every point the isotropy subgroup of B is trivial. Thus B acts simply transitively on each orbit. Without loss of generality we can therefore assume from now on that S is solvable and acts simply transitively on each orbit of $\mathfrak{F}$. This shows that the classification up to isometric congruence of all homogeneous foliations of codimension one on M is equivalent to the classification up to orbit equivalence of all $(n-1)$-dimensional connected solvable subgroups of G^o acting freely on M.

Since the Lie algebra $\mathfrak{s}$ of S is solvable, there exists a maximal solvable subalgebra of $\mathfrak{g}$ containing $\mathfrak{s}$. The maximal solvable subalgebras of real semisimple Lie algebras have been classified by Mostow [21]. Any maximal solvable subalgebra of $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The Cartan subalgebra $\mathfrak{h}$ is the direct sum of its toroidal part $\mathfrak{t}$ and its vector part $\mathfrak{a}$. Here, the toroidal part $\mathfrak{t}$ consists of all $X \in \mathfrak{h}$ for which the eigenvalues of $\mathrm{ad}(X)$ are purely imaginary, and the vector part $\mathfrak{a}$ consists of all $X \in \mathfrak{h}$ for which the eigenvalues of $\mathrm{ad}(X)$ are real. There exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{t} \subset \mathfrak{k}$ and $\mathfrak{a} \subset \mathfrak{p}$. Since $\mathfrak{a}$ is abelian it induces a root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda$ of $\mathfrak{g}$, where Σ denotes the set of roots in the dual vector space $\mathfrak{a}^*$. An element $H \in \mathfrak{a}$ is called regular if $\alpha(H) \neq 0$ for all $\lambda \in \Sigma$. The set of all regular elements in $\mathfrak{a}$ forms an open and dense subset of $\mathfrak{a}$. We choose a regular element $H \in \mathfrak{a}$ and define $\Sigma^+ = \{\lambda \in \Sigma \mid \lambda(H) > 0\}$ and $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$. Then $\mathfrak{h} + \mathfrak{n}$ is a maximal solvable subalgebra of $\mathfrak{g}$. Conversely, as was shown by Mostow [21], any maximal solvable subalgebra of the real semisimple Lie algebra $\mathfrak{g}$ arises in this way.

Now let $\mathfrak{h} + \mathfrak{n}$ be a maximal solvable subalgebra of $\mathfrak{g}$ that contains $\mathfrak{s}$. Note that for a Cartan subalgebra $\mathfrak{h}$ the vector part $\mathfrak{a}$ is in general not a maximal abelian subspace of $\mathfrak{p}$. If $\mathfrak{a}$ is maximal abelian, then $\mathfrak{h}$ is called a maximally noncompact Cartan subalgebra. In this case Σ is a root system and the above decomposition of $\mathfrak{g}$ is the restricted root space decomposition of $\mathfrak{g}$.

Lemma 5.1. *If $\mathfrak{h} + \mathfrak{n}$ is a maximal solvable subalgebra of $\mathfrak{g}$ with $\mathfrak{s} \subset \mathfrak{h} + \mathfrak{n}$, then $\mathfrak{h}$ is a maximally noncompact Cartan subalgebra of $\mathfrak{g}$.*

Proof. Let HN and AN be the connected subgroup of G^o with Lie algebra $\mathfrak{h} + \mathfrak{n}$ and $\mathfrak{a} + \mathfrak{n}$, respectively. The orbits of the actions of HN and AN on M coincide. Therefore, in order that S has orbits of codimension one, we necessarily need that the dimension of $\mathfrak{a} + \mathfrak{n}$ is at least $n - 1$. Assume that $\mathfrak{h}$ is not maximally noncompact. Then the vector part $\mathfrak{a}$ of $\mathfrak{h}$ is strictly contained in some maximal abelian subspace $\widetilde{\mathfrak{a}}$ in $\mathfrak{p}$. Let $\mathfrak{g} = \widetilde{\mathfrak{g}}_0 \oplus \bigoplus_{\widetilde{\lambda}\in\widetilde{\Sigma}} \widetilde{\mathfrak{g}}_{\widetilde{\lambda}}$ be the restricted root space decomposition of $\mathfrak{g}$ with respect to $\widetilde{\mathfrak{a}}$, where $\widetilde{\Sigma}$ is the corresponding set of restricted roots. By construction, the restriction to $\mathfrak{a}$ of any root in $\widetilde{\Sigma}$ is either trivial or a root in Σ. This implies $\mathfrak{g}_0 = \widetilde{\mathfrak{g}}_0 \oplus \bigoplus_{\widetilde{\lambda}\in\widetilde{\Sigma},\widetilde{\lambda}|\mathfrak{a}=0} \widetilde{\mathfrak{g}}_{\widetilde{\lambda}}$

and $\mathfrak{g}_\lambda = \bigoplus_{\widetilde{\lambda}\in\widetilde{\Sigma},\widetilde{\lambda}|\mathfrak{a}=\lambda} \widetilde{\mathfrak{g}}_{\widetilde{\lambda}}$ for all $\lambda \in \Sigma$. We choose a set of simple roots in $\widetilde{\Sigma}$ such that $\widetilde{\lambda}(H) \geq 0$ for any such simple root $\widetilde{\lambda}$, and define $\widetilde{\mathfrak{n}} = \bigoplus_{\widetilde{\lambda}\in\widetilde{\Sigma}^+} \widetilde{\mathfrak{g}}_{\widetilde{\lambda}}$ where $\widetilde{\Sigma}^+$ denotes the resulting set of positive restricted roots in $\widetilde{\Sigma}$. Clearly, we have $\mathfrak{n} \subset \widetilde{\mathfrak{n}}$. Since $\mathfrak{a}$ is strictly contained in $\widetilde{\mathfrak{a}}$, the inequality $\dim(\mathfrak{a}+\mathfrak{n}) \geq n-1$ and the equality $\dim(\widetilde{\mathfrak{a}} + \widetilde{\mathfrak{n}}) = n$ imply that $\mathfrak{n} = \widetilde{\mathfrak{n}}$ and $\dim \mathfrak{a} = \dim \widetilde{\mathfrak{a}} - 1$. According to Sugiura [24, Theorem 5], the one-dimensional normal space of $\mathfrak{a}$ in $\widetilde{\mathfrak{a}}$ is contained in a wall of a Weyl chamber in $\widetilde{\mathfrak{a}}$. But this implies that there exists a root in $\widetilde{\Sigma}^+$ whose restriction to $\mathfrak{a}$ is trivial, which contradicts the equality $\mathfrak{n} = \widetilde{\mathfrak{n}}$. Thus $\mathfrak{h}$ is a maximally noncompact Cartan subalgebra of $\mathfrak{g}$. q.e.d.

A maximally noncompact Cartan subalgebra $\mathfrak{h}$ of a real semisimple Lie algebra $\mathfrak{g}$ is of the form $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$, where $\mathfrak{a}$ is a maximal abelian subspace in the vector part $\mathfrak{p}$ of some Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{t}$ is a maximal abelian subalgebra in the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Any two maximally noncompact Cartan subalgebras of a real semisimple Lie algebra are conjugate to each other. Thus we have proved:

Proposition 5.2. *Let S be a connected solvable closed subgroup of G^o that acts freely on M with codimension one orbits. Then there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, a restricted root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\lambda\in\Sigma} \mathfrak{g}_\lambda$ with respect to a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, and a set Λ of simple roots in Σ, such that the Lie algebra $\mathfrak{s}$ of S satisfies $\mathfrak{s} \subset \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{t}$ is a maximal abelian subalgebra in the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$ and $\mathfrak{n} = \bigoplus_{\lambda\in\Sigma^+} \mathfrak{g}_\lambda$, where Σ^+ denotes the set of positive restricted roots determined by Λ. In this situation $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ is a Cartan subalgebra and $\mathfrak{h} + \mathfrak{n}$ is a maximal solvable subalgebra of $\mathfrak{g}$.*

We may assume from now on that $\mathfrak{s}$ is contained in a maximal solvable subalgebra of the form $\mathfrak{t} + \mathfrak{a} + \mathfrak{n}$ as described in Proposition 5.2. We denote by $\mathfrak{s}_c$ the image of $\mathfrak{s}$ under the canonical projection from $\mathfrak{t} + \mathfrak{a} + \mathfrak{n}$ onto the compact part $\mathfrak{t}$. Analogously, $\mathfrak{s}_n$ denotes the image of $\mathfrak{s}$ under the canonical projection from $\mathfrak{t} + \mathfrak{a} + \mathfrak{n}$ onto the noncompact part $\mathfrak{a} + \mathfrak{n}$. Since $\dim \mathfrak{s} = n - 1$ and since S acts freely on M we also have $\dim \mathfrak{s}_n = n-1$. Thus there exists a unit vector $\xi \in \mathfrak{a}+\mathfrak{n}$ such that $\mathfrak{s}_n = (\mathfrak{a} + \mathfrak{n}) \ominus \mathbb{R}\xi$ is the orthogonal complement of $\mathbb{R}\xi$ in $\mathfrak{a} + \mathfrak{n}$.

Lemma 5.3. *We have $\xi \in \mathfrak{a}$, or there exists a simple root $\alpha \in \Lambda$ such that $\xi \in \mathbb{R}H_\alpha + \mathfrak{g}_\alpha$.*

Proof. We decompose ξ into $\xi = \xi_0 + \sum_{\lambda\in\Sigma^+} \xi_\lambda$ with $\xi_0 \in \mathfrak{a}$ and $\xi_\lambda \in \mathfrak{g}_\lambda$. We have to show that if $\xi_\lambda \neq 0$ for some $\lambda \in \Sigma^+$, then $\lambda \in \Lambda$,

$\xi_0 \in \mathbb{R}H_\lambda$, and $\xi_\mu = 0$ for all $\mu \in \Sigma^+ \setminus \{\lambda\}$. Assume that $\xi_\lambda \neq 0$ for some $\lambda \in \Sigma^+$. Without loss of generality we can assume that λ is not the sum of two roots in $\{\mu \in \Sigma^+ \mid \xi_\mu \neq 0\}$.

We first show that $\lambda \in \Lambda$. Assume that $\lambda \notin \Lambda$. Consider the natural gradation $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_m$ of $\mathfrak{n}$ generated by $\mathfrak{n}_1 = \bigoplus_{\mu\in\Lambda} \mathfrak{g}_\mu$. Since $\lambda \notin \Lambda$, there exist vectors $X_\mu \in \mathfrak{g}_\mu$ and $Y_\nu \in \mathfrak{g}_\gamma$, where $\mu, \nu \in \Sigma^+$ with $\mu + \nu = \lambda$, such that $\xi_\lambda = \bigoplus_{\mu,\nu\in\Sigma^+,\mu+\nu=\lambda}[X_\mu, Y_\nu]$. The assumption that λ is not the sum of two roots in $\{\mu \in \Sigma^+ \mid \xi_\mu \neq 0\}$ ensures that these vectors X_μ and Y_ν are in $\mathfrak{s}_n$. Thus there exist vectors $S_\mu, T_\nu \in \mathfrak{t}$ such that $S_\mu + X_\mu$ and $T_\nu + Y_\nu$ are in $\mathfrak{s}$. Therefore we also have $[S_\mu, Y_\nu] - [T_\nu, X_\mu] + [X_\mu, Y_\nu] = [S_\mu + X_\mu, T_\nu + Y_\nu] \in \mathfrak{s}$. But since the derived subalgebra of $\mathfrak{t}+\mathfrak{a}+\mathfrak{n}$ is contained in $\mathfrak{n}$, we must have $[\mathfrak{s},\mathfrak{s}] \subset \mathfrak{n}$ and hence $[S_\mu, Y_\nu] - [T_\nu, X_\mu] + [X_\mu, Y_\nu] = [S_\mu + X_\mu, T_\nu + Y_\nu] \in \mathfrak{s}_n$. As $\mathfrak{t}$ is contained in $\mathfrak{m}$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, each element in $\mathfrak{t}$ leaves each root space invariant. Thus $[S_\mu, Y_\nu] \in \mathfrak{g}_\nu \subset \mathfrak{s}_n$ and $[T_\nu, X_\mu] \in \mathfrak{g}_\mu \subset \mathfrak{s}_n$. Altogether this implies that $\xi_\lambda = \bigoplus_{\mu,\nu\in\Sigma^+,\mu+\nu=\lambda}[X_\mu, Y_\nu] = \bigoplus_{\mu,\nu\in\Sigma^+,\mu+\nu=\lambda}([S_\mu + X_\mu, T_\nu + Y_\nu] - [S_\mu, Y_\nu] + [T_\nu, X_\mu]) \in \mathfrak{s}_n$, which is a contradiction to $\xi_\lambda \neq 0$. Thus we conclude that $\lambda \in \Lambda$.

We next show that $\xi_0 \in \mathbb{R}H_\lambda$. Assume that $\xi_0 \notin \mathbb{R}H_\lambda$. Then there exists a vector $H \in \mathfrak{a}$ that is perpendicular to ξ_0 and satisfies $\lambda(H) \neq 0$. Since H is perpendicular to ξ_0, we have $H \in \mathfrak{s}_n$, and hence there exists a vector $S \in \mathfrak{t}$ such that $S + H \in \mathfrak{s}$. The vector $\xi_0 + x\xi_\lambda$ with $x = -|\xi_0|^2/|\xi_\lambda|^2 < 0$ is perpendicular to ξ and hence in $\mathfrak{s}_n$. Thus there exists a vector $T \in \mathfrak{t}$ such that $T + \xi_0 + x\xi_\lambda \in \mathfrak{s}$. We conclude that $x[S, \xi_\lambda] + x\lambda(H)\xi_\lambda = [S + H, T + \xi_0 + x\xi_\lambda] \in \mathfrak{s}_n$. Since $\mathrm{ad}(S)$ is a skewsymmetric transformation, we know that $[S, \xi_\lambda]$ is perpendicular to ξ_λ and hence in $\mathfrak{s}_n$. Since also $x \neq 0$ and $\lambda(H) \neq 0$ we conclude that $\xi_\lambda \in \mathfrak{s}_n$, which is a contradiction. It follows that $\xi_0 \in \mathbb{R}H_\lambda$.

In the next step we show that $\xi_\mu = 0$ for all $\mu \in \Sigma^+ \setminus \{\lambda\}$ (if Σ is a reduced root system) resp. $\mu \in \Sigma^+ \setminus \{\lambda, 2\lambda\}$ (if Σ is not reduced). Assume that there exists such a root μ with $\xi_\mu \neq 0$. Then there exists a vector $H \in \mathfrak{a}$ with $\lambda(H) = 0$ and $\mu(H) \neq 0$. Since $\xi_0 \in \mathbb{R}H_\lambda$ and $\lambda(H) = 0$, we see that H is perpendicular to ξ_0 and hence in $\mathfrak{s}_n$. Thus there exists a vector $S \in \mathfrak{t}$ such that $S+H \in \mathfrak{s}$. The vector $\xi_\lambda + y\xi_\mu$ with $y = -|\xi_\lambda|^2/|\xi_\mu|^2 < 0$ is perpendicular to ξ and hence in $\mathfrak{s}_n$. Thus there exists a vector $T \in \mathfrak{t}$ such that $T + \xi_\lambda + y\xi_\mu \in \mathfrak{s}$. Then $[S, \xi_\lambda] + y[S, \xi_\mu] + \lambda(H)\xi_\lambda + y\mu(H)\xi_\mu = [S + H, T + \xi_\lambda + y\xi_\mu] \in \mathfrak{s}_n$. Since $[S, \xi_\lambda] \in \mathfrak{g}_\lambda$ is perpendicular to ξ_λ and $[S, \xi_\mu] \in \mathfrak{g}_\mu$ is perpendicular to ξ_μ, we see that $[S, \xi_\lambda]$ and $[S, \xi_\mu]$ are in $\mathfrak{s}_n$. Since also $\lambda(H) = 0$, $y \neq 0$ and $\mu(H) \neq 0$, we therefore get $\xi_\mu \in \mathfrak{s}_n$, which is a contradiction. It follows that $\xi_\mu = 0$ for all $\mu \in \Sigma^+ \setminus \{\lambda, 2\lambda\}$.

If Σ is a reduced root system we have just finished the proof of Lemma 5.3. Now let us assume that Σ is not reduced. We have to show that $\xi_{2\lambda} = 0$. Assume that $\xi_{2\lambda} \neq 0$. Then the vectors $H_\lambda + x\xi_\lambda$ and $\xi_\lambda + z\xi_{2\lambda}$ with $x = -\langle H_\lambda, \xi_0\rangle/|\xi_\lambda|^2$ and $z = -|\xi_\lambda|^2/|\xi_{2\lambda}^2| < 0$ are in $\mathfrak{s}_n$. Hence there exist vectors $S, T \in \mathfrak{t}$ such that $S + H_\lambda + x\xi_\lambda$ and $T + \xi_\lambda + z\xi_{2\lambda}$ are in $\mathfrak{s}$. Now we get $[S, \xi_\lambda] + z[S, \xi_{2\lambda}] + |H_\lambda|^2\xi_\lambda + 2z|H_\lambda|^2\xi_{2\lambda} - x[T, \xi_\lambda] = [S + H_\lambda + x\xi_\lambda, T + \xi_\lambda + z\xi_{2\lambda}] \in \mathfrak{s}_n$. Since $[S, \xi_\lambda]$, $[S, \xi_{2\lambda}]$ and $[T, \xi_\lambda]$ are in $\mathfrak{s}_n$, we get that $\xi_\lambda + 2z\xi_{2\lambda} \in \mathfrak{s}_n$. But this implies $0 = \langle \xi, \xi_\lambda + 2z\xi_{2\lambda}\rangle = |\xi_\lambda|^2 + 2z|\xi_{2\lambda}|^2 = -|\xi_\lambda|^2$, which is a contradiction. Thus we must have $\xi_{2\lambda} = 0$, by which the proof of Lemma 5.3 is finished. q.e.d.

Proposition 5.4. *Let $\mathfrak{s}$ be an $(n-1)$-dimensional linear subspace of $\mathfrak{h} + \mathfrak{n}$ with $\mathfrak{s} \cap \mathfrak{t} = 0$.*

(1) *$\mathfrak{s}_n$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$ if and only if there exists a unit vector ξ in $\mathfrak{a}$ or in $\mathbb{R}H_\alpha + \mathfrak{g}_\alpha$ for some simple root $\alpha \in \Lambda$ such that $\mathfrak{s}_n = (\mathfrak{a} + \mathfrak{n}) \ominus \mathbb{R}\xi$;*

(2) *$\mathfrak{s}$ is a subalgebra of $\mathfrak{h} + \mathfrak{n}$ if and only if $\mathfrak{s}_n$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$ and the linear map $L : \mathfrak{s}_n \to \mathfrak{t}$ defined by $LX + X \in \mathfrak{s}$ for all $X \in \mathfrak{s}_n$ satisfies $[LX, Y] + [X, LY] \in \mathfrak{s}_n$ and $L([LX, Y] + [X, LY] + [X, Y]) = 0$ for all $X, Y \in \mathfrak{s}_n$.*

Proof. If $\mathfrak{s}_n$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$, then it is also a subalgebra of $\mathfrak{h} + \mathfrak{n}$, and the noncompact part $(\mathfrak{s}_n)_n$ of $\mathfrak{s}_n$ is $\mathfrak{s}_n$ itself. It follows from Lemma 5.3 that there exists a unit vector ξ in $\mathfrak{a}$ or in $\mathbb{R}H_\alpha + \mathfrak{g}_\alpha$ for some simple root $\alpha \in \Lambda$ such that $\mathfrak{s}_n = (\mathfrak{a} + \mathfrak{n}) \ominus \mathbb{R}\xi$. Conversely, since the derived subalgebra of $\mathfrak{a} + \mathfrak{n}$ is $\mathfrak{n}$, and considering the gradation $\mathfrak{n} = \mathfrak{n}_1 \oplus \cdots \oplus \mathfrak{n}_m$, where $\mathfrak{n}_1 = \sum_{\alpha\in\Lambda} \mathfrak{g}_\alpha$, it is easy to see that $(\mathfrak{a} + \mathfrak{n}) \ominus \mathbb{R}\xi$ is a Lie subalgebra of $\mathfrak{a} + \mathfrak{n}$ for any such vector ξ. This proves Part (1).

Since $\mathfrak{s} \cap \mathfrak{t} = 0$, it is clear that $L : \mathfrak{s}_n \to \mathfrak{t}$ is a well-defined linear map and that $\mathfrak{s} = \{LX + X \mid X \in \mathfrak{s}_n\}$. The subspace $\mathfrak{s}$ is a subalgebra of $\mathfrak{h} + \mathfrak{n}$ if and only if $[LX, Y] + [X, LY] + [X, Y] = [LX + X, LY + Y] \in \mathfrak{s}$ for all $X, Y \in \mathfrak{s}_n$. But this is equivalent to saying that for all $X, Y \in \mathfrak{s}_n$ there exists a vector $Z \in \mathfrak{s}_n$ such that $[LX, Y] + [X, LY] + [X, Y] = LZ + Z$. The vector on the left-hand side of this equation is in $\mathfrak{n}$. Thus $\mathfrak{s}$ is a subalgebra of $\mathfrak{h} + \mathfrak{n}$ if and only if for all $X, Y \in \mathfrak{s}_n$ there exists a vector $Z \in \mathfrak{s}_n$ with $LZ = 0$ such that $[LX, Y] + [X, LY] + [X, Y] = Z$. If $\mathfrak{s}$ is a subalgebra of $\mathfrak{h} + \mathfrak{n}$, then $\mathfrak{s}_n$ is a subalgebra of $\mathfrak{a} + \mathfrak{n}$ according to

Lemma 5.3 and Part (1) of this proposition. Thus, if $\mathfrak{s}$ is a subalgebra of $\mathfrak{h}+\mathfrak{n}$ then $[LX,Y]+[X,LY] \in \mathfrak{s}_n$ and $L([LX,Y]+[X,LY]+[X,Y]) = 0$ for all $X, Y \in \mathfrak{s}_n$. Conversely, if $\mathfrak{s}_n$ is a subalgebra of $\mathfrak{a}+\mathfrak{n}$ and if L satisfies the previous two conditions, the last equivalence statement implies that $\mathfrak{s}$ is a subalgebra. q.e.d.

According to Proposition 5.4, $\mathfrak{s}_n$ is a subalgebra of $\mathfrak{a}+\mathfrak{n}$. Let S_n be the connected subgroup of AN with Lie algebra $\mathfrak{s}_n$. Every element in $s \in S$ can be written in the form $s = tan$ with $t \in T$, $a \in A$ and $n \in N$, where T, A, N is the connected Lie subgroup of G^o with Lie algebra $\mathfrak{t}, \mathfrak{a}, \mathfrak{n}$, respectively. Based on Lemma 5.3 we distinguish three cases, namely

Case 1. $\xi \in \mathfrak{a}$.

Case 2. $\xi \in \mathfrak{g}_\alpha$ for some simple root $\alpha \in \Lambda$.

Case 3. $\xi = aH_\alpha + X_\alpha$ for some $0 \neq a \in \mathbb{R}$, $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ and some simple root $\alpha \in \Lambda$.

Case 1. $\xi \in \mathfrak{a}$. Then we have $\mathfrak{s}_n = (\mathfrak{a} \ominus \mathbb{R}\xi) \oplus \mathfrak{n}$, where $\mathfrak{a} \ominus \mathbb{R}\xi$ denotes the orthogonal complement of $\mathbb{R}\xi$ in $\mathfrak{a}$. Since $\mathfrak{s}_c$ is contained in the centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k}$, it is clear that $\mathfrak{s}_c$ leaves each root space $\mathfrak{g}_\lambda$, $\lambda \in \Sigma$, invariant. Thus we have $[\mathfrak{s}_c, \mathfrak{n}] \subset \mathfrak{n}$. Moreover, $\mathfrak{s}_c \subset \mathfrak{t}$ and $\mathfrak{t}+\mathfrak{a}$ is abelian, which shows that $[\mathfrak{s}_c, \mathfrak{a} \ominus \mathbb{R}\xi] = 0$. Therefore, if $tan \in S$, we have $at = ta$ and $tn = n't$ for some $n' \in N$. We thus get $tan \cdot o = atn \cdot o = an't \cdot o = an' \cdot o \in S_n \cdot o$, which shows that $S \cdot o \subset S_n \cdot o$. Since both orbits $S \cdot o$ and $S_n \cdot o$ have the same dimension and are connected and complete, we conclude $S \cdot o = S_n \cdot o$. This shows that the actions of S and S_n are orbit-equivalent, and hence the foliation $\mathfrak{F}$ is just the foliation $\mathfrak{F}_\xi$.

Case 2. $\xi \in \mathfrak{g}_\alpha$ for some simple root $\alpha \in \Lambda$. In this case we have $\mathfrak{s}_n = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathbb{R}\xi) = \mathfrak{a} \oplus (\mathfrak{g}_\alpha \ominus \mathbb{R}\xi) \oplus \bigoplus_{\lambda \in \Sigma^+ \setminus \{\alpha\}} \mathfrak{g}_\lambda$. Let $\lambda \in \Sigma^+ \setminus \{\alpha\}$ and $X \in \mathfrak{g}_\lambda$. Using Proposition 5.4 we get $0 = L([LH_\lambda, X] + [H_\lambda, LX] + [H_\lambda, X]) = L(\{\mathrm{ad}(LH_\lambda) + \lambda(H_\lambda)I_\lambda\}X)$, where I_λ denotes the identity transformation on $\mathfrak{g}_\lambda$. But $\mathrm{ad}(LH_\lambda)$ is a skewsymmetric transformation on $\mathfrak{g}_\lambda$ and $\lambda(H_\lambda)$ is nonzero. Thus the transformation $\mathrm{ad}(LH_\lambda) + \lambda(H_\lambda)I_\lambda : \mathfrak{g}_\lambda \to \mathfrak{g}_\lambda$ is an isomorphism, and we conclude that $L\mathfrak{g}_\lambda = 0$. Next, for $X \in \mathfrak{g}_\alpha \ominus \mathbb{R}\xi$ we have $[LH_\alpha, X] = [LH_\alpha, X] + [H_\alpha, LX] \in \mathfrak{s}_n \cap \mathfrak{g}_\alpha$ by means of Proposition 5.4, and hence $[LH_\alpha, X] \in \mathfrak{g}_\alpha \ominus \mathbb{R}\xi$. An analogous argument as above implies $L(\mathfrak{g}_\alpha \ominus \mathbb{R}\xi) = 0$. Altogether we conclude that $L(\mathfrak{n} \ominus \mathbb{R}\xi) = 0$. Therefore we have $\mathfrak{s}_c = L\mathfrak{s}_n = L\mathfrak{a}$, and using Proposition 5.4 we get $[\mathfrak{s}_c, \mathfrak{n} \ominus \mathbb{R}\xi] = [L\mathfrak{a}, \mathfrak{n} \ominus \mathbb{R}\xi] \subset$

$\mathfrak{s}_n \cap \mathfrak{n} = \mathfrak{n} \ominus \mathbb{R}\xi$. As in Case 1 we can now deduce that $S \cdot o = S_n \cdot o$ and hence $\mathfrak{F}$ is equal to $\mathfrak{F}_\xi$.

Case 3. $\xi = aH_\alpha + X_\alpha$ for some $0 \neq a \in \mathbb{R}$, $0 \neq X_\alpha \in \mathfrak{g}_\alpha$ and some simple root $\alpha \in \Lambda$. We put $X = X_\alpha/|X_\alpha|$, define a subalgebra $\mathfrak{s}_X$ of $\mathfrak{a}+\mathfrak{n}$ by $\mathfrak{s}_X = \mathfrak{a} \oplus (\mathfrak{n} \ominus \mathbb{R}X)$, and denote by S_X the corresponding connected Lie subgroup of AN. As was shown in the proof of Lemma 4.2 there exists an isometry $g \in AN$ with $\mathrm{Ad}(g)(\mathfrak{s}_n) = \mathfrak{s}_X$. The foliations on M that are induced by the actions of S and $I_g(S)$ are obviously isometrically congruent. We will now show that the foliations that are induced by the actions of $I_g(S)$ and S_X are equal. Clearly, since $g \in AN$, we have $\mathrm{Ad}(g)\mathfrak{a} \subset \mathfrak{a} + \mathfrak{n}$, $\mathrm{Ad}(g)\mathfrak{n} = \mathfrak{n}$ and $\mathrm{Ad}(g)(\mathfrak{a} + \mathfrak{n}) = \mathfrak{a} + \mathfrak{n}$. Furthermore, $\mathrm{Ad}(g)(\mathfrak{h} + \mathfrak{n}) = \mathrm{Ad}(g)\mathfrak{h} + \mathfrak{n}$ is a maximal solvable subalgebra of $\mathfrak{g}$ and $\mathrm{Ad}(g)\mathfrak{h} = \mathrm{Ad}(g)\mathfrak{t} + \mathrm{Ad}(g)\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. We define a linear map $L_g : \mathfrak{s}_X \to \mathrm{Ad}(g)\mathfrak{t}$ by $L_g = \mathrm{Ad}(g)L\mathrm{Ad}(g)^{-1}$, where $L : \mathfrak{s}_n \to \mathfrak{t}$ is the linear map defined by $LY + Y \in \mathfrak{s}$ for all $Y \in \mathfrak{s}_n$. It can be easily verified that L_g satisfies the two conditions in Proposition 5.4. For the Lie algebra $\mathrm{Ad}(g)\mathfrak{s}$ of $I_g(S)$ we then get $\mathrm{Ad}(g)\mathfrak{s} = \{\mathrm{Ad}(g)(LY + Y) \mid Y \in \mathfrak{s}_n\} = \{L_gZ + Z \mid Z \in \mathrm{Ad}(g)\mathfrak{s}_n = \mathfrak{s}_X\}$. This implies that $(\mathrm{Ad}(g)\mathfrak{s})_n = \mathfrak{s}_X$, and from Case 2 we conclude that the foliations that are induced by the actions of $I_g(S)$ and S_X are equal. Altogether we now see that $\mathfrak{F}$ is isometrically congruent to $\mathfrak{F}_X$. This concludes Case 3.

Since all Cartan decompositions $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$, all maximal abelian subspaces $\mathfrak{a}$ of $\mathfrak{p}$, and all choices of sets Λ of simple roots in the corresponding set Σ of restricted roots are conjugate by inner automorphisms of $\mathfrak{g}$ we thus have proved:

Theorem 5.5. *Let $\mathfrak{F}$ be a homogeneous foliation of codimension one on M. Then $\mathfrak{F}$ is isometrically congruent to one of the model foliations $\mathfrak{F}_\ell$ or $\mathfrak{F}_i$.*

The main theorem now follows from Theorems 3.5, 4.8 and 5.5.

References

[1] D.V. Alekseevskii, *The conjugacy of polar factorizations of Lie groups*, Math. USSR-Sb. **13** (1971) 12–24, MR 43 #3395, Zbl 0239.22013.

[2] S. Araki, *On root systems and an infinitesimal classification of irreducible symmetric spaces*, J. Math. Osaka City Univ. **13** (1962) 1–34, MR 27 #3743, Zbl 0123.03002.

[3] L. Bérard-Bergery, *Sur de nouvelles variétés riemanniennes d'Einstein*, Inst. Élie Cartan **6** (1982) 1–60, MR 85b:53048, Zbl 0544.53038.

[4] J. Berndt, *Homogeneous hypersurfaces in hyperbolic spaces*, Math. Z. **229** (1998) 589–600, MR 2001c:53065, Zbl 0929.53025.

[5] J. Berndt & M. Brück, *Cohomogeneity one actions on hyperbolic spaces*, J. Reine Angew. Math. **541** (2001) 209–235, MR 2002j:53059.

[6] J. Berndt & H. Tamaru, *Cohomogeneity one actions on noncompact symmetric spaces with a totally geodesic singular orbit*, to appear in Tôhoku Math. J..

[7] E. Cartan, *Familles de surfaces isoparamétriques dans les espaces à courbure constante*, Ann. Mat. pura appl. IV. s. **17** (1938) 177–191, Zbl 0020.06505.

[8] J.E. D'Atri, *Certain isoparametric families of hypersurfaces in symmetric spaces*, J. Differential Geom. **14** (1979) 21–40, MR 81g:53041, Zbl 0459.53040.

[9] D. Ferus, H. Karcher & H.-F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z. **177** (1981) 479–502, MR 83k:53075, Zbl 0443.53037.

[10] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York-London, 1978, MR 80k:53081, Zbl 0451.53038.

[11] S. Helgason, *Geometric analysis on symmetric spaces*, Mathematical Surveys and Monographs, **39**, American Mathematical Society, Providence, RI, 1994, MR 96h:43009, Zbl 0809.53057.

[12] W.-Y. Hsiang & H.B. Lawson, Jr., *Minimal submanifolds of low cohomogeneity*, J. Differential Geom. **5** (1971) 1–38, MR 45 #7645, Zbl 0219.53045.

[13] K. Iwata, *Classification of compact transformation groups on cohomology quaternion projective spaces with codimension one orbits*, Osaka J. Math. **15** (1978) 475–508, MR 80k:57068, Zbl 0415.57026.

[14] K. Iwata, *Compact transformation groups on rational cohomology Cayley projective planes*, Tôhoku Math. J. (2) **33** (1981) 429–442, MR 83h:57047, Zbl 0506.57024.

[15] S. Kaneyuki & H. Asano, *Graded Lie algebras and generalized Jordan triple systems*, Nagoya Math. J. **112** (1988) 81–115, MR 90d:17026, Zbl 0699.17021.

[16] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. **354** (2002) 571–612, MR 2002g:53091.

[17] A.W. Knapp, *Lie groups beyond an introduction*, Birkhäuser, Boston MA, 1996, MR 98b:22002, Zbl 0862.22006.

[18] T. Levi-Civita, *Famiglie di superficie isoparametriche nell'ordinario spazio euclideo*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (6) **26** (1937) 355–362, Zbl 0018.08702.

[19] A. Malcev, *On the theory of the Lie groups in the large*, Rec. Math. [Mat. Sbornik] N.S. **16** (1945) 163–190, MR 7,115c, Zbl 0061.04601; Correction: ibid. **19** (1946) 523–524, MR 8,312c, Zbl 0061.04602.

[20] P.S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math. (2) **65** (1957) 447–455, MR 19,44b, Zbl 0080.16702; Errata: ibid. **66** (1957) 589, MR 20 #2395.

[21] G.D. Mostow, *On maximal subgroups of real Lie groups*, Ann. of Math. (2) **74** (1961) 503–517, MR 26 #256, Zbl 0109.02301.

[22] B. Rosenfeld, *Geometry of Lie groups*, Kluwer Academic Publishers, Dordrecht, 1997, MR 98i:53002, Zbl 0867.53002.

[23] B. Segre, *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (6) **27** (1938) 203–207, Zbl 0019.18403.

[24] M. Sugiura, *Conjugate classes of Cartan subalgebras in real semisimple Lie algebras*, J. Math. Soc. Japan **11** (1959) 374–434, MR 26 #3827, Zbl 0204.04201; Correction: ibid. **23** (1971) 379–383, MR 44 #1772, Zbl 0215.09505.

[25] R. Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973) 495–506, MR 49 #1433, Zbl 0274.53062.

[26] R. Takagi & S. Yorozu, *Minimal foliations on Lie groups*, Tôhoku Math. J. (2) **36** (1984) 541–554, MR 86b:53028, Zbl 0559.53018.

[27] F. Uchida, *Classification of compact transformation groups on cohomology complex projective spaces with codimension one orbits*, Japan. J. Math. (N.S.) **3** (1977) 141–189, MR 80b:57036, Zbl 0374.57009.

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